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Noncommutative Geometry and Physics: Renormalisation, Motives, Index Theory

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2010 Mathematics Subject Classification (primary; secondary): 58B34, 11M55, 11G09 11M06, 11M32, 47G30, 81Q30, 81T15, 17A30; 16T30, 18D50, 46L80, 46L87, 19K33, 19K56, 58J42, 58J20, 81Q30

Key words: Arithmetic geometry, motives, noncommutative geometry, index theory, K-theory, pre-Lie algebras, Feynman integrals, zeta functions, Birkhoff–Hopf factorisation, renormalisation

ISBN 978-3-03719-008-1

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

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Typeset using the author's TEX files: I. Zimmermann, Freiburg
Printed in Germany

9 8 7 6 5 4 3 2 1

Introduction

This volume is a collection of expository articles on mathematical topics of current interest that have been partly stimulated or influenced by interactions with physical theory. Some of these articles are based on lectures given at various venues over the last few years, revised and written especially for this volume. They cover a diverse range of mathematical topics stemming from different parts of physics.

The major underlying theme of the volume involves the interactions between physics and number theory. This theme is manifested in two ways, first through the study of Feynman integrals and renormalisation theory, and second, through the application of methods from quantum statistical mechanics. In the former, the work of Bogoliubov–Parasuik–Hepp–Zimmermann (BPHZ) on renormalisation theory gave a method for step by step control of divergences and of their regularisation in Feynman's approach to perturbative quantum field theory. Already, in the evaluation of Feynman integrals, the occurrence of multizeta values hinted at deeper mathematical connections. This deeper underlying structure was found by Alain Connes and Dirk Kreimer in the form of associating to each renormalisable quantum field theory a Hopf algebra that provided a systematic understanding of the BPHZ procedure in terms of a Birkhoff factorisation in a Lie group associated to the Hopf algebra. A contemporary view of this fundamental work is provided here by Christoph Bergbauer. The latter strand, the relationship between statistical mechanics and number theory, began much earlier through the work of Jean-Benoît Bost and Alain Connes. Bost–Connes introduced a quantum statistical mechanical dynamical system that captures information on the primes and on the Riemann zeta function. They determined the equilibrium states of their model (the so-called Kubo–Martin–Schwinger (KMS) states) and its phase transitions. Subsequent extensions of this basic idea to number theoretic questions resulted in the theory of 'endomotives'. Motives also arise in the study of Feynman integrals and hence we have provided here an introduction to these ideas in the Ramdorai–Plazas–Marcolli article.

We now give a brief overview of the contents of this volume.

Bergbauer's article discusses Feynman integrals, regularization and renormalization following the algebraic approach to the Feynman rules developed by Bloch, Connes, Esnault, Kreimer, and others. It reviews several renormalization methods found in the literature from a single point of view using resolution of singularities, and includes a discussion of the motivic nature of Feynman integrals.

Motives are explained in much greater detail in the article of Sujatha Ramdorai and Jorge Plazas. The construction of the category of pure motives is explained here starting from the category of smooth projective varieties. They also survey the theory of endomotives developed by D. C. Cisinski and G. Tabuada, which links the theory of motives to the quantum statistical mechanical techniques that connect number theory and noncommutative geometry. The appendix to this article, contributed by Matilde Marcolli, elaborates these latter ideas providing a useful introduction to, and summary of, the role of KMS states. Also described is the view of motives that arises from noncommutative geometry along with a detailed account of the interweaving of number theory with statistical mechanics, noncommutative geometry and endomotives.

Hopf algebras associated to rooted trees are known to systemetise the combinatorics of Feynman graphs through the work of Connes–Kreimer. Dominique Manchon describes another algebraic object associated to rooted trees, namely pre-Lie algebras. His article reviews the basic theory of pre-Lie algebras and also describes how they arise from operads. Their relation to other algebraic structures and their application to numerical analysis are described as well.

Multiple zeta values arise from the evaluation of Feynman integrals. Sylvie Paycha's contribution discusses generalisations of renormalised multiple zeta values at nonpositive integers. As with some of the other articles, the exposition is partly inspired by renormalised Feynman integrals in physics using pseudodifferential symbols.

Zeta residues and pseudodifferential analysis, both of which play a role in earlier articles, also arise in index theory. In turn, index theory is well known to play a role in gauge field theories via the study of anomalies. These are described mathematically by the families index theorem. Under the influence of Alain Connes and others, a noncommutative approach to index theory, partly inspired by quantum theory, has emerged over the last two decades. This noncommutative index theory is described in the contribution of Alan Carey, John Phillips and Adam Rennie, beginning with a discussion of classical index theorems from a noncommutative point of view. This is followed by a review of K-theory and cyclic cohomology that culminates in a description of the local index formula in noncommutative geometry. The article concludes with an example that underlies recent applications of noncommutative geometry to Mumford curves.

All of the articles in this volume were partly inspired by a program in Number Theory and Physics held at the Erwin Schrödinger Institute from March 2 to April 18, 2009. The editor would like to thank ESI for its hospitality and support for the Number Theory and Physics program, Arthur Greenspoon for his enthusiastic assistance with proof-reading and Steve Rosenberg and Irene Zimmermann for their very valuable contributions to the preparation of this volume.

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Notes on Feynman integrals and renormalization

Christoph Bergbauer

Contents

1 Introduction

In recent years there has been a growing interest in Feynman graphs and their integrals.

Physicists use Feynman graphs and the associated integrals in order to compute certain experimentally measurable quantities out of quantum field theories. The problem is that there are conceptual difficulties in the definition of interacting quantum field theories in four dimensions. The good thing is that nonetheless the Feynman graph formalism is very successful in the sense that the quantities obtained from it match with the quantities obtained in experiment extremely well. Feynman graphs are interpreted as elements of a perturbation theory, i.e., as an expansion of an (interesting) interacting quantum field theory in the neighborhood of a (simple) free quantum field theory. One therefore hopes that a better understanding of Feynman graphs and their integrals could eventually lead to a better understanding of the true nature of quantum field theories, and contribute to some of the longstanding open questions in the field.

A Feynman graph is simply a finite graph, to which one associates a certain integral: the integrand depends on the quantum field theory in question, but in the simplest case it is just the inverse of a direct product of rank 4 quadratic forms, one for each edge of the graph, restricted to a real linear subspace determined by the topology of the graph.

For a general graph, there is currently no canonical way of solving this integral analytically. However, in this simple case where the integrand is algebraic, one can be convinced to regard the integral as a period of a mixed motive, another notion which is not rigorously defined as of today. All the Feynman periods that have been computed so far, are rational linear combinations of multiple zeta values, which are known to be periods of mixed Tate motives, a simpler and better understood kind of motives. A stunning theorem of Belkale and Brosnan however indicates that this is possibly a coincidence due to the relatively small number of Feynman periods known today: They showed that in fact any algebraic variety defined over $\mathbb{Z}$ is related to a Feynman graph hypersurface (the Feynman period is one period of the motive of this hypersurface) in a quite obscure way.

The purpose of this article is to review selected aspects of Feynman graphs, Feynman integrals and renormalization in order to discuss some of the recent work by Bloch, Esnault, Kreimer and others on the motivic nature of these integrals. It is based on public lectures given at the ESI in March 2009, at the DESY and IHES in April and June 2009, and several informal lectures in a local seminar in Mainz in fall and winter 2009. I would like to thank the other participants for their lectures and discussions.

Much of my approach is centered around the notion of renormalization, which seems crucial for a deeper understanding of Quantum Field Theory. No claim of originality is made except for Section 3.2 and parts of the surrounding sections, which is a review of my own research with R. Brunetti and D. Kreimer [10], and Section 3.6 which contains new results.

This article is not meant to be a complete and up to date survey by any means. In particular, several recent developments in the area, for example the work of Brown [24], [26], Aluffi and Marcolli [3], [2], [1], Doryn and Schnetz [35], [76], and the theory of Connes and Marcolli [32], [67] are not covered here.

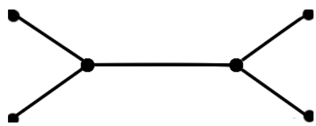
Acknowledgements. I thank S. Müller-Stach, R. Brunetti, S. Bloch, M. Kontsevich, P. Brosnan, E. Vogt, C. Lange, A. Usnich, T. Ledwig, F. Brown and especially D. Kreimer for discussions on the subject of this article. I would like to thank the ESI and the organizers of the spring 2009 program on number theory and physics for hospitality during the month of March 2009, and the IHES for hospitality in January and February 2010. My research is funded by the SFB 45 of the Deutsche Forschungsgemeinschaft.

2 Feynman graphs and Feynman integrals

For the purpose of this article, a Feynman graph is simply a finite connected multigraph where "multi" means that there may be several, parallel edges between vertices. Loops, i.e., edges connecting to the same vertex at both ends, are not allowed in this article. Roughly, physicists think of edges as virtual particles and of vertices as interactions between the virtual particles corresponding to the adjacent edges.

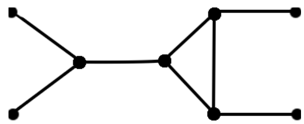
If one has to consider several types of particles, one has several types (colors, shapes etc.) of edges.

Here is an example of a Feynman graph:

 (1)

This Feynman graph describes a theoretical process within a scattering experiment: a pair of particles annihilates into a third, intermediate, particle, and this third particle then decays into the two outgoing particles on the right.

This Feynman graph (and the probability amplitude assigned to it) make sense only as a single term in a first order approximation. In order to compute the scattering cross section, one will have to sum over arbitrarily complicated Feynman graphs with four fixed external edges, and in this sum an infinity of graphs with cycles will occur, for example:



In this article we will be concerned only with Feynman graphs containing cycles, and I will simply omit the external edges that correspond to the (asymptotic) incoming and outgoing physical particles of a scattering experiment.

I will come back to the physical interpretation in greater detail in Section 2.3.

2.1 Feynman rules. Feynman graphs are not only a nice tool for drawing complex interactions of virtual particles; they also provide a recipe to compute the probability that certain scattering processes occur. The theoretical reason for this will be explained later, but to state it very briefly, a Feynman graph is regarded as a *label* for a term in a perturbative expansion of this probability amplitude. This term in this expansion is called a *Feynman integral*, but at this point one must be careful with the word integral because of reasons of convergence.

Definition 2.1. An *integral* is a pair (A, u) where A is an open subset of some $\mathbb{R}^n$ or $\mathbb{R}^n_{\geq 0}$, and u a distribution in $A \cap (\mathbb{R}^n \setminus \bigcup H_i)$, where H_i are affine subspaces.

A distribution in X is a continuous linear functional on the space of *compactly supported* test functions $C_0^\infty(X)$ with the usual topology. Locally integrable functions (that is, functions integrable on compact subsets) define distributions in an obvious way. Let us denote by $\underline{1}_A$ the characteristic function of A in $\mathbb{R}^n$. It is certainly not a test function unless A is compact, but for suitable u (decays rapidly enough at ∞) we may evaluate u against $\underline{1}_A$. We write $u[f]$ for the distribution applied to the test function f. If u is given by a locally integrable function, we may also write $\int u(x)f(x)dx$.

If u is given by a function which is integrable over all of A, then (A, u) can be associated with the usual integral $\int_A u(x)dx = u[\underline{1}_A]$. Feynman integrals however are very often divergent: this means by definition that $\int_A u(x)dx$ is divergent, and this can either result from problems with local integrability at the H_i or lack of integrability at ∞ away from the H_i (if A is unbounded), or both. (A more unified point of view would be to start with a $\mathbb{P}^n$ instead of $\mathbb{R}^n$ in order to have the divergence at ∞ as a divergence at the hyperplane H_∞ at ∞, but I will not exploit this here.)

A basic example of such a divergent integral is the pair $A = \mathbb{R} \setminus \{0\}$ and $u(x) = |x|^{-1}$. The function u is locally integrable inside A, hence a distribution in A. But it is neither integrable as $|x| \to \infty$, nor locally integrable at $\{0\}$. We will see in a moment that the divergent Feynman integrals to be defined are higher-dimensional generalizations of this example, with an interesting arrangement of the H_i.

In this section I will introduce three kinds of Feynman integrals associated to a given graph: a position space, a momentum space and a parametric integral. The first two are related by a Fourier transform, and the second and the third by a change of variables. It may be useful to emphasize at this point that all three amount to the same "value" once they are properly and consistently renormalized (a notion that I will introduce in the next section).

The following approach, which I learned from S. Bloch [15], [14], is quite powerful when one wants to understand the various Feynman rules from a common point of view. It is based on the idea that a Feynman graph first defines a point configuration in some $\mathbb{R}^n$, and it is only this point configuration which determines the Feynman integral via the Feynman rules.

Let Γ be a Feynman graph with set of edges $E(\Gamma)$ and set of vertices $V(\Gamma)$. A subgraph γ has by definition the same vertex set $V(\gamma) = V(\Gamma)$ but $E(\gamma) \subseteq E(\Gamma)$. Impose temporarily an orientation of the edges, such that every edge has an incoming $v_{\mathrm{e,in}}$ and an outgoing vertex $v_{\mathrm{e,out}}$. Since we do not allow loops, the two are different. Set $(v : e) = 1$ if v is the outgoing vertex of e, $(v : e) = -1$ if v is the incoming vertex and e, and $(v : e) = 0$ otherwise. Let $\mathcal{M} = \mathbb{R}^d$, where $d \in 2 + 2\mathbb{N}$, called *space-time*, with Euclidean metric $|\cdot|$. We will mostly consider the case where $d = 4$, but it is useful to see the explicit dependence on d in the formulas.

All the information of Γ is encoded in the map

$$\mathbb{Z}^{E(\Gamma)} \xrightarrow{\partial} \mathbb{Z}^{V(\Gamma)}$$

sending an edge $e \in E(\Gamma)$ to $\partial(e) = \sum_{v \in V(\Gamma)} (v : e)v = v_{\mathrm{e,out}} - v_{\mathrm{e,in}}$. This is nothing but the chain complex for the oriented simplicial homology of the 1-dimensional simplicial complex Γ, and it is a standard construction to build from this map ∂ an exact sequence

$$0 \to H_1(\Gamma; \mathbb{Z}) \to \mathbb{Z}^{E(\Gamma)} \xrightarrow{\partial} \mathbb{Z}^{V(\Gamma)} \to H_0(\Gamma; \mathbb{Z}) \to 0. \tag{2}$$

From this one obtains two inclusions of free abelian groups into $\mathbb{Z}^{E(\Gamma)}$:

$$i_\Gamma \colon H_1(\Gamma; \mathbb{Z}) \hookrightarrow \mathbb{Z}^{E(\Gamma)};$$

the second one is obtained by dualizing

$$j_\Gamma \colon \mathbb{Z}^{V(\Gamma)\vee}/H^0(\Gamma; \mathbb{Z}) \overset{\partial^\vee}{\hookrightarrow} \mathbb{Z}^{E(\Gamma)\vee}.$$

Here, and generally whenever a basis is fixed, we can canonically identify free abelian groups with their duals.

All this can be tensored with $\mathbb{R}$, and we get inclusions i_Γ, j_Γ of vector spaces into another vector space with a *fixed basis*. If one then replaces any $\mathbb{R}^n$ by $\mathcal{M}^n$ and denotes

$i_\Gamma^{\oplus d} = (i_\Gamma, \ldots, i_\Gamma)$, $j_\Gamma^{\oplus d} = (j_\Gamma, \ldots, j_\Gamma)$, then two types of Feynman integrals (A, u) are defined as follows:

$$A_M = H_1(\Gamma; \mathbb{R})^d, \qquad u_\Gamma^M = (i_\Gamma^{\oplus d})^* u_{0,M}^{\otimes |E(\Gamma)|},$$
$$A_P = \mathcal{M}^{V(\Gamma)^\vee} / H^0(\Gamma; \mathbb{R})^d, \quad u_\Gamma^P = (j_\Gamma^{\oplus d})^* u_{0,P}^{\otimes |E(\Gamma)|}.$$

The distributions $u_{0,M}, u_{0,P} \in \mathcal{D}'(\mathcal{M})$ therein are called *momentum space,* resp. *position space propagators*. Several examples of propagators and how they are related will be discussed in the next section, but for a first reading

$$u_{0,M}(p) = \frac{1}{|p|^2}, \quad u_{0,P}(x) = \frac{1}{|x|^{d-2}},$$

inverse powers of a rank d quadratic form. As announced earlier, the pullbacks $(i_\Gamma^{\oplus d})^* u_{0,M}^{\otimes |E(\Gamma)|}$ and $(j_\Gamma^{\oplus d})^* u_{0,M}^{\otimes |E(\Gamma)|}$ are only defined as distributions outside certain affine spaces H_i, that is, for test functions supported on compact subsets which do not meet these H_i.

The map

$$\Gamma \mapsto (A_M, u_\Gamma^M)$$

is called *momentum space Feynman rules*, and the map

$$\Gamma \mapsto (A_P, u_\Gamma^P)$$

is called *position space Feynman rules*.

Usually, in the physics literature, the restriction to the subspace is imposed by multiplying the direct product of propagators with several delta distributions which are interpreted as "momentum conservation" at each vertex in the momentum space picture, and dually "translation invariance" in the position space case.

In position space, it is immediately seen that

$$u_\Gamma^P = (j_\Gamma^{\oplus d})^* u_{0,P}^{\otimes |E(\Gamma)|} = \pi_* \prod_{e \in E(\Gamma)} u_{0,P}(x_{\text{e,out}} - x_{\text{e,in}}),$$

where π_* means pushforward along the projection $\pi : M^{V(\Gamma)^\vee} \to M^{V(\Gamma)^\vee} / H^0(\Gamma)^d$, see [10].

In momentum space, things are a bit more complicated.

Definition 2.2. A connected graph Γ is called *core* if $\operatorname{rk} H_1(\Gamma \setminus \{e\}) < \operatorname{rk} H_1(\Gamma)$ for all $e \in E(\Gamma)$.

By Euler's formula (which follows from the exactness of (2))

$$\operatorname{rk} H_1(\Gamma) - |E(\Gamma)| + |V(\Gamma)| - \operatorname{rk} H_0(\Gamma) = 0,$$

it is equivalent for a connected graph Γ to be core and to be one-particle-irreducible (1PI), a physicists' notion: Γ is one-particle-irreducible if removing an edge does not disconnect Γ.

Let now Γ be connected and core; then

$$u_\Gamma^M = (i_\Gamma^{\oplus d})^* u_{0,M}^{\otimes |E(\Gamma)|} = \prod_{e \in E(\Gamma)} u_{0,M}(p_e) \prod_{v \in V(\Gamma)} \delta_0\Big(\sum_{e \in E(\Gamma)} (v:e) p_e\Big).$$

This is simply because $\operatorname{im} i_\Gamma = \ker \partial$, and because for

$$\partial\Big(\sum_{e \in E(\Gamma)} p_e e\Big) = \sum_{e \in E(\Gamma)} p_e \sum_v (v:e) v = 0$$

it is necessary that

$$\sum_{e \in E(\Gamma)} (v:e) p_e = 0 \quad \text{for all } v \in V(\Gamma).$$

(The requirement that Γ be core is really needed here because otherwise certain $e \in E(\Gamma)$ would never show up in a cycle, and hence would be missing inside the delta function.)

Moreover, one can define a version of u_Γ^M which depends additionally on *external momenta* $P_v \in \mathcal{M}$, one for each $v \in V(\Gamma)$, up to momentum conservation for each component $\sum_{v \in C} P_v = 0$:

$$U_\Gamma^M(\{P_v\}_{v \in V(\Gamma)}) = \prod_{e \in E(\Gamma)} u_{0,M}(p_e) \prod_{v \in V(\Gamma)} \delta_0(P_v + \sum_{e \in E(\Gamma)} (v:e) p_e).$$

By a slight abuse of notation I keep the P_v, $v \in V(\Gamma)$, as coordinate vectors for $\mathcal{M}^{|V(\Gamma)|}/H^0(\Gamma, \mathbb{R})^d = A_P$ and identify distributions on A_P with distributions on $\mathcal{M}^{|V(\Gamma)|}$ that are multiples of $\prod_C \delta_0(\sum_{v \in C} P_v)$.

U_Γ^M is now a distribution on a subset of $A_P \times A_M$, and

$$U_\Gamma^M|_{P_v = 0, v \in V(\Gamma)} = u_\Gamma^M.$$

The vectors $P_v \in A_P$ determine a shift of the linear subspace $A_M = H_1(\Gamma; \mathbb{R})^{\oplus d} \hookrightarrow \mathcal{M}^{|E(\Gamma)|}$ to an affine one. Usually all but a few of the P_v are set to zero, namely all but those which correspond to the incoming or outgoing particles of an experiment (see Section 2.3).

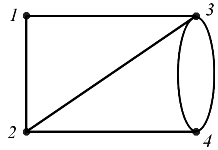
The relation between the momentum space and position space distributions is then a Fourier duality. I denote by $\mathcal{F}$ the Fourier transform.

Proposition 2.1. *If the basic propagators are Fourier-dual ($\mathcal{F} u_{0,P} = u_{0,M}$), as is the case for $u_{0,M}(p) = \frac{1}{|p|^2}$ and $u_{0,P}(x) = \frac{1}{|x|^{d-2}}$, then*

$$(U_\Gamma^M[\underline{1}_{A_M}])(\{P_v\}) = \mathcal{F} u_\Gamma^P$$

where only the (internal) momenta of A_M are integrated out; and this holds up to convergence issues only, i.e., in the sense of Definition 2.1.

For example, the graph



gives rise to

$$u^M_{\Gamma_3} = u^2_{0,M}(p_1)u_{0,M}(p_2)u_{0,M}(p_1 + p_2)u_{0,M}(p_3)u_{0,M}(p_2 + p_3),$$
$$u^P_{\Gamma_3} = u_{0,P}(x_1 - x_2)u_{0,P}(x_1 - x_3)u_{0,P}(x_2 - x_3)u_{0,P}(x_2 - x_4)u^2_{0,P}(x_3 - x_4),$$

where $p^i_1, \dots, p^i_3$, $i = 0, \dots, d - 1$ is a basis of coordinates for A_M and $x^i_1, \dots, x^i_4$, $i = 0, \dots, d - 1$ is a basis of coordinates for $\mathcal{M}^{V(\Gamma_3)^\vee}$ (If Γ is connected, dividing by $H^0(\Gamma; \mathbb{R})^d$ takes care of the joint (diagonal) translations by $\mathcal{M}$ and, as previously, instead of writing distributions on $\mathcal{M}^{V(\Gamma)^\vee}/H^0(\Gamma; \mathbb{R})^d$, I take the liberty of writing translation-invariant distributions on $\mathcal{M}^{V(\Gamma)^\vee}$).

Finally, the case of external momenta:

$$\begin{aligned} U^M_{\Gamma_3}(P_1, P_2, 0, P_4) = u_{0,M}(p_1)&u_{0,M}(p_1 + P_1)u_{0,M}(p_2) \\ &\cdot u_{0,M}(p_1 + p_2 + P_1 + P_2)u_{0,M}(p_3) \\ &\cdot u_{0,M}(p_2 + p_3 + P_4)\delta_0(P_1 + P_2 + P_4). \end{aligned} \tag{3}$$

I set one of the external momenta, P_3, to zero in order to have a constant number of 4 adjacent (internal and external) momenta at each vertex: P_1 is the sum of two external momenta at the vertex 1 (see Section 2.3 for the reason).

We will come back to the question of the affine subspaces H_i where u^M_Γ, resp. u^P_Γ is not defined in the section about renormalization.

In general, following [15], Section 2, a configuration is just an inclusion of a vector space W into another vector space $\mathbb{R}^E$ with fixed basis E: The dual basis vectors $e^\vee$, $e \in E$ determine linear forms on W, and those linear forms (or dually the linear hyperplanes annihilated by them) are the "points" of the configuration in the usual sense. By the above construction, any such configuration, plus the choice of a propagator, defines an integral.

If the configuration comes from a Feynman graph, the integral is called a *Feynman integral*.

2.2 Parametric representation. Integrals can be rewritten in many ways, using linearity of the integrand, of the domain, change of variables and Stokes' theorem, and possibly a number of other tricks.

For many purposes it will be useful to have a version of the Feynman rules with a domain A which is much lower-dimensional than in the previous section but has boundaries and corners. The first part of the basic trick here is to rewrite the propagator

$$u_0 = \int_0^\infty \exp(-a_e u_0^{-1}) da_e$$

(whenever the choice of propagator allows this inversion; $u_0(p) = \frac{1}{|p|^2}$ certainly does), introducing a new coordinate $a_e \in \mathbb{R}_{\geq 0}$ for each edge $e \in E(\Gamma)$. From this one has a distribution

$$\bigotimes_{e \in E(\Gamma)} \exp(-a_e u_0^{-1}(p_e)) = \exp\big(-\sum_{e \in E(\Gamma)} a_e u_0^{-1}(p_e)\big) \tag{4}$$

in $(\mathcal{M} \times \mathbb{R}_{\geq 0})^{|E(\Gamma)|}$. From now on I assume $u_0(p) = \frac{1}{|p|^2}$. Suppose that $i \colon W \hookrightarrow \mathbb{R}^{|E(\Gamma)|}$ is an inclusion. Once a basis of W is fixed, the linear form $e^\vee i$ is a row vector in W and its transpose $(e^\vee i)^t$ a column vector in W. The product $(e^\vee i)^t(e^\vee i)$ is then a $\dim W$-square matrix. Pulling back (4) along an inclusion $i^{\oplus d} \colon W \hookrightarrow \mathcal{M}^{|E(\Gamma)|}$ (such as $i^{\oplus d} = i_\Gamma^{\oplus d}$ or $i^{\oplus d} = j_\Gamma^{\oplus d}$) means imposing linear relations on the p_e. These relations can be transposed onto the a_e: after integrating Gaussian integrals over W (this is the second part of the trick) and a change of variables, one is left with the distribution

$$u_\Gamma^S(\{a_e\}) = \big(\det \sum_{e \in E(\Gamma)} a_e (e^\vee i)^t (e^\vee i)\big)^{-d/2}$$

on $A_S = \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$ except for certain intersections H_i of coordinate hyperplanes $\{a_e = 0\}$. I discarded a multiplicative constant $C_\Gamma = (2\pi)^{d \dim W/2}$ which does not depend on the topology of the graph.

Suppose that $d = 4$. Depending on whether $i = i_\Gamma$ or j_Γ there is a momentum space and a position space version of this trick. The two are dual to each other in the following sense:

$$\det \sum_{e \in E(\Gamma)} a_e (e^\vee i_\Gamma)^t (e^\vee i_\Gamma) = \big(\prod_{e \in E(\Gamma)} a_e\big) \det \sum_{e \in E(\Gamma)} a_e^{-1} (e^\vee j_\Gamma)^t (e^\vee j_\Gamma)$$

See [15], Proposition 1.6, for a proof. In this article, we will only consider the momentum space version, where $i = i_\Gamma$. The map

$$\Gamma \mapsto (A_S, u_\Gamma^S)$$

with $i = i_\Gamma$ is called the *Schwinger* or *parametric Feynman rules*. Just as in the previous section, there is also a version with external momenta, which I just quote from [47], [16], [14],

$$U_\Gamma^S(\{a_e\}, \{P_v\}) = \frac{\exp(-(N^{-1}P)^t P)}{(\det \sum_{e \in E(\Gamma)} a_e (e^\vee i_\Gamma)^t (e^\vee i_\Gamma))^2}$$

where

$$N = \sum_{e \in E(\Gamma)} a_e^{-1} (e^\vee j_\Gamma)^t (e^\vee j_\Gamma),$$

a $d(|V(\Gamma)| - \dim H_0(\Gamma; \mathbb{R}))$-square matrix.

The determinant

$$\Psi_\Gamma(a_e) = \det \sum_{e\in E(\Gamma)} a_e (e^\vee i_\Gamma)^t (e^\vee i_\Gamma)$$

is a very special polynomial in the a_e. It is called the *first graph polynomial*, *Kirchhoff polynomial* or *Symanzik polynomial*. It can be rewritten

$$\Psi_\Gamma(a_e) = \sum_{T \text{ sf of } \Gamma} \prod_{e\notin E(T)} a_e \tag{5}$$

as a sum over *spanning forests* (sf) T of Γ: A spanning forest is a subgraph $E(T) \subseteq E(\Gamma)$ such that the map $\partial|_{\mathbb{R}^{E(T)}} : \mathbb{R}^{E(T)} \to \mathbb{R}^{V(\Gamma)}/H_0(\Gamma;\mathbb{R})$ is an isomorphism; in other words, a subgraph without cycles that has exactly the same components as Γ. (In the special case where Γ is connected, a spanning forest is called a *spanning tree* (st) and is characterized by being connected as well and having no cycles.)

For the *second graph polynomial* Φ_Γ, which is a polynomial in the a_e and a quadratic form in the P_v, let us assume for simplicity that Γ is connected. Then

$$\Phi_\Gamma(a_e, P_v) = \Psi_\Gamma \cdot (N^{-1}P)^t P = \sum_{T \text{ st of } \Gamma} \sum_{e_0\in E(T)} P_1^t P_2 a_{e_0} \prod_{e\notin E(T)} a_e$$

where $P_A = \sum_{v\in C_A} P_v$ is the sum of momenta in the first connected component C_A and $P_B = \sum_{v\in C_B} P_v$ the sum of momenta in the second connected component C_B of the graph $E(T)\setminus\{e_0\}$ (which has exactly two components since T is a spanning tree). See [15], [16], [47] for proofs.

Here is a simple example: If

$$\Gamma_2 =$$

then

$$\Psi_{\Gamma_2} = a_1 + a_2, \quad \Phi_{\Gamma_2} = P_1^2 a_1 a_2$$

and

$$U_\Gamma^S = \frac{\exp\left(-P_1^2 \frac{a_1 a_2}{a_1+a_2}\right)}{(a_1+a_2)^2}.$$

All this holds if $u_{0,M} = \frac{1}{|p|^2}$. If $u_{0,M} = \frac{1}{|p|^2+m^2}$ then

$$U_\Gamma^S = \exp\left(-m^2 \sum_{e\in E(\Gamma)} a_e\right) U_\Gamma^S|_{m=0}.$$

2.3 The origin of Feynman graphs in physics. Before we continue with a closer analysis of the divergence locus of these Feynman integrals, it will be useful to have at least a basic understanding of why they were introduced in physics. See [88], [28], [54], [42], [45], [73], [87], [33], for a general exposition, and I follow in particular [73], [42], [45] in this section. Quantum Field Theory is a theory of particles which obey the basic principles of quantum mechanics and special relativity at the same time. Special relativity is essentially the study of mechanics covariant under the Poincaré group

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$$

(where $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3)^+$ is the universal double cover of the identity component $\mathrm{O}(1,3)^+$ of $\mathrm{O}(1,3)$). In other words, $\mathcal{P}$ is the double cover of the group of (space- and time-) orientation-preserving isometries of Minkowski space-time $\mathbb{R}^{1,3}$ (I assume $d = 4$ in this section).

On the other hand, quantum mechanics always comes with a Hilbert space, a vacuum vector, and operators on the Hilbert space.

By definition, a *single particle* is then an irreducible unitary representation of $\mathcal{P}$ on some Hilbert space H_1. Those have been classified by Wigner according to the joint spectrum of $P = (P_0, \dots, P_3)$, the vector of infinitesimal generators of the translations. Its joint spectrum (as a subset of $\mathbb{R}^{1,3}$) is either one of the following $\mathrm{SL}(2,\mathbb{C})$-orbits: the hyperboloids (mass shells)

$$S_\pm(m) = \{(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m^2,\ p^0 \gtrless 0\} \subset \mathbb{R}^{1,3} \quad (m > 0),$$

and the forward and backward light cones $S_\pm(0) \subset \mathbb{R}^{1,3}$ $(m = 0)$. (There are two more degenerate cases, for example $m < 0$ which I do not consider further.) This gives a basic distinction between massive $(m > 0)$ and massless particles $(m = 0)$. For a finer classification, one looks at the stabilizer subgroups G_p at $p \in S_\pm(m)$. If $m > 0$, then $G_p \cong \mathrm{SU}(2,\mathbb{C})$, if $m = 0$, then G_p is the double cover of the group of isometries of the Euclidean plane. In any case, the G_p are pairwise conjugate in $\mathrm{SL}(2,\mathbb{C})$ and

$$H_1 = \int_\oplus H^p \, d\Omega_m(p),$$

where the H^p are pairwise isomorphic and carry an irreducible representation of G_p. By $d\Omega_m$ I denote the unique $\mathrm{SL}(2,\mathbb{C})$-invariant measure on $S_\pm$. The second classifying parameter is then an invariant of the representation of G_p on H^p: In the case where $m > 0$ and $G_p \cong \mathrm{SU}(2,\mathbb{C})$, one can take the dimension: $H^p \cong \mathbb{C}^{2s+1}$, and $s \in \mathbb{N}/2$ is called the *spin*. If $m = 0$, G_p acts on $\mathbb{C}$ by mapping a rotation by the angle ϕ around the origin to $e^{in\phi} \in \mathbb{C}^*$, and $n/2$ is called the *helicity* (again I dismiss a few cases that are of no physical interest).

In summary, one identifies a single particle of mass m and spin s or helicity n with the Hilbert space

$$H_1 \cong L^2(S_\pm(m), d\Omega_m) \otimes \mathbb{C}^{2s+1} \quad \text{resp.} \quad L^2(S_\pm(0), d\Omega_0),$$

and a *state* of the given particle is an element of the projectivized Hilbert space $\mathbb{P}H_1$.

Quantum field theories describe many-particle systems, and particles can be generated and annihilated. A general result in quantum field theory, the Spin-Statistics theorem [62, 55], tells us that systems of particles with integer spin obey Bose (symmetric) statistics while those with half-integer spin obey Fermi (antisymmetric) statistics. We stick to the case of $s = 0$, and most of the time even $m = 0$, $n = 0$, (which can be considered as the limit $m \to 0$ of the massive case) in this article.

The Hilbert space of infinitely many non-interacting particles of the same type, called Fock space, is then the symmetric tensor algebra

$$H = \mathrm{Sym}\, H_1 = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n\, H_1$$

of H_1 (for fermions, one would use the exterior algebra instead). $\mathcal{P}$ acts on H in the obvious way; denote the representation by U, and $\Omega = 1 \in \mathbb{C} = \mathrm{Sym}^0\, H_1 \subset H$ is called *vacuum vector*.

Particles are created and annihilated as follows: If $f \in \mathcal{D}(\mathbb{R}^{1,3})$ is a test function, then $\hat{f} = \mathcal{F} f|_{S_\pm(m)} \in H_1$ (the Fourier transform is taken with respect to the Minkowski metric) and

$$\begin{aligned} a^\dagger[f]\colon\ & \mathrm{Sym}^{n-1}\, H_1 \to \mathrm{Sym}^n\, H_1 \colon \\ & \Phi(p_1, \ldots, p_{n-1}) \mapsto \sum_{i=1}^{n} \hat{f}(p_i)\Phi(p_1, \ldots, \widehat{p_i}, \ldots, p_n), \\ a[f]\colon\ & \mathrm{Sym}^{n+1}\, H_1 \to \mathrm{Sym}^n\, H_1 \colon \\ & \Phi(p_1, \ldots, p_{n+1}) \mapsto \int_{S_\pm(m)} \overline{\hat{f}(p)}\Phi(p, p_1, \ldots, p_n)\, d\Omega_m(p) \end{aligned}$$

define operator-on-H-valued distributions $f \mapsto a^\dagger[f]$, $f \mapsto a[f]$ on $\mathbb{R}^{1,3}$. The operator $a^\dagger[f]$ creates a particle in the state $\hat{f}$ (i.e., with smeared momentum $\hat{f}$), and $a[f]$ annihilates one.

The sum

$$\phi = a + a^\dagger$$

is called the *field*. It is the quantized version of the classical field, which is a C^∞ function on Minkowski space. The field ϕ on the other hand is an operator-valued distribution on Minkowski space. It satisfies the Klein–Gordon equation

$$(\Box + m^2)\phi = 0 \tag{6}$$

($\Box$ is the Laplacian of $\mathbb{R}^{1,3}$) which is the Euler-Lagrange equation for the classical Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{7}$$

The tuple (H, U, ϕ, Ω) and one extra datum which I omit here for simplicity is what is usually referred to as a quantum field theory satisfying the Wightman axioms [78]. The axioms require certain $\mathcal{P}$-equivariance, continuity and *locality* conditions.

The tuple I have constructed (called the *free* scalar field theory) is a very well understood one because (6), resp. the Lagrangian (7), are very simple indeed. As soon as one attempts to construct a quantum field theory $(H_I, U_I, \phi_I, \Omega_I)$ for an interacting Lagrangian (which looks more like a piece of the Lagrangian of the Standard Model) such as

$$\mathcal{L}_0 + \mathcal{L}_I = \frac{1}{2}(\partial_\mu \phi_I)^2 - \frac{1}{2}m^2\phi_I^2 + \lambda\phi_I^n \tag{8}$$

($n \geq 3$, $\lambda \in \mathbb{R}$ is called the *coupling constant*) one runs into serious trouble. In this rigorous framework the existence and construction of non-trivial interacting quantum field theories in four dimensions is as of today an unsolved problem, although there is an enormous number of important partial results; see for example [75].

However, one can expand quantities of the interacting quantum field theory as a formal power series in λ with coefficients quantities of the free field theory, and hope that the series has a positive radius of convergence. This is called the *perturbative expansion*. In general the power series has radius of convergence 0, but due to some non-analytic effects which I do not discuss further, the first terms in the expansion do give a very good approximation to the experimentally observed quantities for many important interacting theories (this is the reason why quantum field theories have played such a prominent role in the physics of the last 50 years).

I will devote the remainder of this section to a sketch of this perturbative expansion, and how the Feynman integrals introduced in the previous section arise there.

By Wightman's reconstruction theorem [78], a quantum field theory

$$(H_I, U_I, \phi_I, \Omega_I)$$

is uniquely determined by and can be reconstructed from the *Wightman functions* (distributions) $w_n^I = \langle \Omega_I, \phi_I(x_1)\ldots\phi_I(x_n)\Omega_I\rangle$. Similar quantities are the *time-ordered Wightman functions*

$$t_n^I = \langle \Omega_I, T(\phi_I(x_1)\ldots\phi_I(x_n))\Omega_I\rangle,$$

which appear directly in scattering theory. If one knows all the t_n^I, one can compute all scattering cross-sections. The symbol T denotes time-ordering:

$$\begin{aligned} T(\psi_1(x_1)\psi_2(x_2)) &= \psi_1(x_1)\psi_2(x_2) \quad \text{if } x_1^0 \geq x_2^0, \\ &= \psi_2(x_2)\psi_1(x_1) \quad \text{if } x_2^0 > x_1^0 \end{aligned}$$

for operator-valued distributions ψ_1, ψ_2.

For the free field theory, all the w_n and t_n are well understood, in particular

$$\begin{aligned} t_2(x_1, x_2) &= \langle \Omega, T(\phi(x_1)\phi(x_2))\Omega\rangle \\ &= \mathcal{F}^{-1}\frac{i}{(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 - m^2 + i\epsilon}, \end{aligned}$$

where the Fourier transform is taken with respect to the difference coordinates $x_1 - x_2$ (the t_n are translation-invariant). t_2 is a particular fundamental solution of equation (6) called the *propagator*. By a technique called *Wick rotation*, one can go back and forth between Minkowski space $\mathbb{R}^{1,3}$ and Euclidean space $\mathbb{R}^4$ [71], [48], turning Lorentz squares $(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$ into Euclidean squares $-|p|^2$, and the Minkowski space propagator t_2 into the distribution $u_{0,P} = \mathcal{F}^{-1}\frac{1}{|p|^2+m^2}$ introduced in the previous sections. In the massless case $m = 0$, we have $u_{0,P} = u_{0,M} = \frac{1}{|x|^2}$ if $d = 4$.

From the usual physics axioms for scattering theory and on a purely symbolic level, Gell-Mann and Low's formula relates the interacting t_n^I with vacuum expectation values $\langle \Omega, T(\dots)\Omega\rangle$ of time-ordered products of powers of the *free* fields,

$$t_n^I(x_1,\dots,x_n) = \sum_{k=0}^{\infty} \frac{i^k}{k!} \int \langle \Omega, T(\phi(x_1)\dots\phi(x_n)\mathcal{L}_I^0(y_1)\dots\mathcal{L}_I^0(y_k))\Omega\rangle d^4y_1\dots d^4y_k, \tag{9}$$

as a formal power series in λ. I denote $\mathcal{L}_I^0 = \mathcal{L}_I|_{\phi_I\to\phi} = \lambda\phi^n$. There is a subtle point here in defining powers of ϕ as operator-valued distributions. The solution is called *Wick (or normal ordered) powers*: in the expansion of $\phi^n = (a + a^\dagger)^n$, all $a^\dagger$ are moved to the left of the a such that no monomials containing $aa^\dagger$ in this order appear. Consequently $\langle\Omega, W(\dots)\Omega\rangle = 0$ for any normal ordered operator $W(\dots)$. Time- and normal ordered products are related by what is called Wick's Theorem:

$$\begin{aligned}
T(\phi(x_1)\phi(x_2)) &= W(\phi(x_1)\phi(x_2)) + t_2(x_1,x_2),\\
T(\phi(x_1)\dots\phi(x_3)) &= W(\phi(x_1)\phi(x_2)\phi(x_3)) + W(\phi(x_1))t_2(x_2,x_3)\\
&\quad + W(\phi(x_2))t_2(x_1,x_3) + W(\phi(x_3))t_2(x_1,x_2),\\
T(\phi(x_1)\dots\phi(x_4)) &= W(\phi(x_1)\dots\phi(x_4)) + W(\phi(x_1)\phi(x_2))t_2(x_3,x_4) + \cdots\\
&\quad \cdots + t_2(x_1,x_2)t_2(x_3,x_4) + \cdots\\
&\ \vdots
\end{aligned}$$

Now within the free field theory, the $\langle\Omega, T(\dots)\Omega\rangle$ are well understood. It follows from the arguments above that $\langle\Omega, T(\dots)\Omega\rangle$ is a polynomial in the t_2; more precisely,

$$\langle\Omega, T(\phi^{n_1}\dots\phi^{n_k})\Omega\rangle = \sum_{\Gamma} c_\Gamma \pi^* u_\Gamma^P, \tag{10}$$

where the sum is over all Feynman graphs Γ with k vertices such that the ith vertex has degree n_i, and where u_Γ^P is defined as in the previous sections, c_Γ is a combinatorial symmetry factor, and $u_{0,P}(x) = t_2(x,0)$ up to a Wick rotation.

For example, the graph (1) arises in

$$\langle\Omega, T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^3(y_1)\phi^3(y_2))\Omega\rangle,$$

a second-order (in λ) contribution to t_4^I of the theory where $\mathcal{L}_I = \lambda\phi_I^3$. The points $x_1, \ldots, x_4$ are external and y_1, y_2 the internal ones.

Generally, if one uses (10) for (9) then one gets Feynman graphs with n external vertices of degree 1. The external edges, i.e., edges leading to those n vertices, appear simply as tensor factors, and can be omitted (amputated) in a first discussion. In this way we are left with the graphs considered in the previous section.

It follows in particular that only Feynman graphs with vertices of degree n appear from the Lagrangian (8). Note that whereas external physical particles are always on-shell (i.e., their momentum supported on $S_\pm$), the internal virtual particles are integrated over all of momentum space in the Gell-Mann–Low formula.

In summary, the perturbative expansion of an interacting quantum field theory (whose existence let alone construction in the sense of the Wightman axioms is an unsolved problem) provides a power series approximation in the coupling constant to the bona fide interacting functions t_n^I. The coefficients are sums of Feynman integrals which are composed of elements of the free theory alone.

3 Regularization and renormalization

The Feynman integrals introduced so far are generally divergent integrals. At first sight it seems to be a disturbing feature of a quantum field theory that it produces divergent integrals in the course of calculations, but a closer look reveals that this impression is wrong: it is only a naive misinterpretation of perturbation theory that makes us think this way.

Key to this is the insight that single Feynman graphs are really about virtual particles, and their parameters, for example their masses, have no real physical meaning. They have to be *renormalized*. In this way the divergences are compensated by so-called counterterms in the Lagrangian of the theory which provide some kind of dynamical contribution to these parameters [28]. I will not make further use of this physical interpretation but only consider the mathematical aspects. If the divergences can be compensated by adjusting only a finite number of parameters in the Lagrangian (i.e., by leaving the form of the Lagrangian invariant and not adding an infinity of new terms to it) the theory is called renormalizable.

An important and somewhat nontrivial, but fortunately solved [19], [46], [90], [38], [53], [29], [30], problem is to find a way to organize this correspondence between removing divergences and compensating counterterms in the Lagrangian for arbitrarily complicated graphs. Since the terms in the Lagrangian are local terms, that is, polynomials in the field and its derivatives, a necessary criterion for this is the so-called *locality of counterterms*: if one has a way of removing divergences such that the correction terms are local ones, then this is a good indication that they fit into the Lagrangian in the first place.

Regularization on the other hand is the physics term used for a variety of methods of writing the divergent integral or integrand as the limit of a holomorphic family of convergent integrals or integrands, say over a punctured disk. Sometimes also the integrand is fixed, and the domain of integration varies holomorphically, say over the punctured disk. We will see a number of such regularizations in the remainder of this article.

3.1 Position space. In position space, the renormalization problem has been known for a long time to be an *extension problem of distributions* [19], [38]. This follows already from our description in Section 2, but it will be useful to have a closer look at the problem. Recall the position space Feynman distribution

$$u_\Gamma^P = (j_\Gamma^{\oplus d})^* u_{0,P}^{\otimes |E(\Gamma)|}$$

is defined only as a distribution on $A_P = \mathcal{M}^{|V(\Gamma)|\vee}/H^0(\Gamma;\mathbb{R})^{\oplus d}$ minus certain affine (in this case even linear) subspaces. Suppose, for example, that

$$\Gamma_2 =$$


with $u_{\Gamma_2}^P = \frac{1}{|x|^{2d-4}}$. If f is a non-negative test function supported in a ball $N = \{|x| \le \epsilon\}$ around 0.

$$u_{\Gamma_2}^P[f] = \int_N f(x) u_{\Gamma_2}^P(x) dx \ge \min_{x\in N} f(x) \int d\Omega \int_0^\epsilon \frac{dr^{d-1}}{r^{2d-4}}.$$

If $d-1-(2d-4) \le -1$, that is, $d \ge 4$, the integral will be divergent at 0 and $u_{\Gamma_2}^P$ not defined on test functions supported at 0. This is the very nature of ultraviolet (i.e., short-distance) divergences. On the other hand, divergences as some position-space coordinates go to ∞ are called infrared (long-distance) divergences. We will be concerned with ultraviolet divergences in this article.

For simplicity we restrict ourselves to graphs with at most logarithmic divergences throughout the rest of the article, that is, $d \operatorname{rk} H_1(\gamma) \ge 2|E(\gamma)|$ for all subgraphs $E(\gamma) \subseteq E(\Gamma)$. A subgraph γ where equality holds is called *divergent*. A detailed power-counting analysis, carried out in [10], shows that u_Γ^P is only defined as a distribution inside

$$A_P^\circ = A_P \setminus \bigcup_{\substack{E(\gamma)\subseteq E(\Gamma)\\ d \operatorname{rk} H_1(\gamma)=2|E(\gamma)|}} \bigcap_{e\in E(\gamma)} \pi D_e, \tag{11}$$

where $D_e = \{x_{\mathrm{e,out}} - x_{\mathrm{e,in}} = 0\}$. The singular support (the locus where u_Γ^P is not smooth) is

$$\operatorname{sing\,supp} u_\Gamma^P = A_P^\circ \cap \bigcup_{e\in E(\Gamma)} \pi D_e.$$

An extension of u_Γ^P from A_P° to A_P is called a *renormalization* provided it satisfies certain consistency conditions to be discussed later.

In the traditional literature, which dates back to a central paper of Epstein and Glaser [38], an extension of u_Γ^P from A_P° to all of A_P was obtained inductively, by starting with the case of two vertices, and embedding the solution (extension) for this case into the three, four, etc. vertex case using a partition of unity. In this way, in each step only one extension onto a single point, say 0, is necessary, a well-understood problem with a finite-dimensional space of degrees of freedom: two extensions differ by a distribution supported at this point 0, and the difference is therefore, by elementary considerations, of the form $\sum_{|\alpha|\leq n} c_\alpha \partial^\alpha \delta_0$ with $c_\alpha \in \mathbb{C}$. Some of these parameters c_α are fixed by physical requirements such as probability conservation, Lorentz and gauge invariance, and more generally the requirement that certain differential equations be satisfied by the extended distributions. But even after these constants are fixed, there are degrees of freedom left, and various groups act on the space of possible extensions, which are collectively called the *renormalization group*. For the at most logarithmic graphs considered in this article, $n = 0$ and only one constant c_0 needs to be fixed in each step.

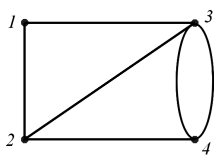
3.2 Resolution of singularities. The singularities, divergences and extensions (renormalizations) of the Feynman distribution u_Γ^P are best understood using a resolution of singularities [10]. The Fulton–MacPherson compactification [43] introduced in a quantum field theory context by Kontsevich [51], [49] and Axelrod and Singer [6] serves as a universal smooth model where all position space Feynman distributions can be renormalized. In [10], a graph-specific De Concini–Procesi wonderful model [34] was used, in order to elaborate the striking match between De Concini and Procesi's notions of building set, nested set and notions found in Quantum Field Theory. No matter which smooth model is chosen, one is led to a smooth manifold Y and a proper surjective map, in fact a composition of blowups,

$$\beta : Y \to A_P,$$

which is a diffeomorphism on $\beta^{-1}(A_P^\circ)$ but where $\beta^{-1}(A_P \setminus A_P^\circ)$ is (the real locus of) a divisor with normal crossings.

Instead of the nonorientable smooth manifold Y one can also find an orientable manifold with corners Y' and β a composition of real spherical blowups as in [6]. In my figures, the blowups are spherical because they are easier to draw, but in the text they are projective.

Here is an example: If Γ_3 is again the graph

$\Gamma_3 =$  (12)

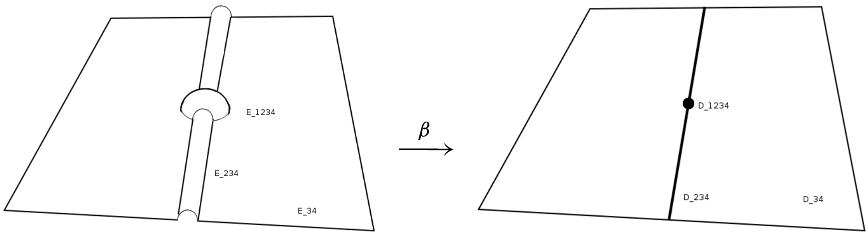
and $d = 4$, then by (11) the locus where there are non-integrable singularities is

$$D_{1234} \subset D_{234} \subset D_{34},$$

where $D_{1234} = D_{12} \cap D_{13} \cap D_{14}$, $D_{234} = D_{23} \cap D_{24}$. In A_P, πD_{1234} is a point, πD_{234} is 4-dimensional and πD_{24} is 8-dimensional. Blowing up something means replacing it by its projectivized normal bundle. The map β is composed of three maps

$$Y = Y_{34} \xrightarrow{\beta_3} Y_{234} \xrightarrow{\beta_2} Y_{1234} \xrightarrow{\beta_1} A_P$$

where β_1 blows up D_{1234}, β_2 blows up the strict transform of D_{234}, and β_3 blows up the strict transform of D_{34}.



Now $u^P_{\Gamma_3}$ can be pulled back along β (because of lack of orientability of Y, it will become a distribution density). In a clever choice of local coordinates, for example

$$\begin{aligned}
y^0_1 &= x^0_1 - x^0_2,\\
y^0_2 &= (x^0_2 - x^0_3)/(x^0_1 - x^0_2),\\
y^0_3 &= (x^0_3 - x^0_4)/(x^0_2 - x^0_3),\\
y^i_1 &= (x^i_1 - x^i_2)/(x^0_1 - x^0_2),\\
y^i_2 &= (x^i_2 - x^i_3)/(x^0_2 - x^0_3),\\
y^i_3 &= (x^i_3 - x^i_4)/(x^0_3 - x^0_4),
\end{aligned}$$

one has

$$w^P_{\Gamma_3} = \beta^* u^P_{\Gamma_3} = \frac{f^P_{\Gamma_3}}{|y^0_1 y^0_2 y^0_3|}, \tag{13}$$

where $f^P_{\Gamma_3}$ is a locally integrable density which is even C^∞ in the coordinates y^0_1, y^0_2, y^0_3. The divergence is therefore isolated in the denominator, and only in three directions: y^0_1, y^0_2 and y^0_3. The first is the local coordinate transversal to the exceptional divisor $\mathcal{E}_{1234}$ of the blowup of D_{1234}, the second transversal to the exceptional divisor $\mathcal{E}_{234}$ of the blowup of D_{234}, and the third transversal to the exceptional divisor $\mathcal{E}_{34}$ of the blowup of D_{34} (the difference between $\mathcal{E}_{34}$ and D_{34} is not seen in the figure for of dimensional reasons).

For a general graph Γ, the total exceptional divisor $\mathcal{E} = \beta^{-1}(A_P \setminus A_P^\circ)$ has normal crossings and the irreducible components $\mathcal{E}_\gamma$ are indexed by connected divergent (consequently core) irreducible subgraphs γ. Moreover,

$$\mathcal{E}_{\gamma_1} \cap \cdots \cap \mathcal{E}_{\gamma_k} \neq \emptyset \iff \text{the } \gamma_i \text{ are nested},$$

where nested means each pair is either disjoint or one contained in the other. See [10] for the general result and more details.

Inspired by old papers of Atiyah [5], Bernstein and Gelfand [12] we used $(u_\Gamma^P)^s$, where s in a complex number in a punctured neighborhood of 1, as a regularization [10]. Similarly, since the propagator $u_{0,P}(x) = \frac{1}{|x|^{d-2}}$ depends on the dimension, one can also consider u_Γ^P with d in a punctured complex neighborhood of 4 as a regularization but I will not pursue this here.

Definition 3.1. A connected graph Γ is called primitive if

$$d \operatorname{rk} H_1(\gamma) = 2|E(\gamma)| \iff E(\gamma) = E(\Gamma)$$

for all subgraphs $E(\gamma) \subseteq E(\Gamma)$.

For a primitive graph Γ_p, only the single point $0 \in A_P$ needs to be blown up, and the pullback along β yields in suitable local coordinates ($y_1^0 = x_1^0 - x_2^0$, $y_i^j = (x_i^j - x_{i+1}^j)/(x_1^0 - x_2^0)$ otherwise)

$$\beta^* u_{\Gamma_p}^P = \frac{f_{\Gamma_p}}{|y_1^0|},$$

where f_{Γ_p} is a locally integrable distribution density constant in the y_1^0-direction. Let $d_\Gamma = d(|V(\Gamma_p)| - 1)$. Consequently

$$\beta^*(u_{\Gamma_p}^P)^s = \frac{f_{\Gamma_p}^s}{|y_1^0|^{d_{\Gamma_p} s - (d_{\Gamma_p} - 1)}}.$$

It is well known that the distribution-valued function $\frac{1}{|x|^s}$ can be analytically continued in a punctured neighborhood of $s = 1$, with a simple pole at $s = 1$. The residue of this pole is δ_0:

$$\frac{1}{|x|^s} = \frac{\delta_0}{s-1} + |x|_{\text{fin}}^s,$$

$$|x|_{\text{fin}}^s[f] = \int_{-1}^{1} |x|^s (f(x) - f(0))dx + \int_{\mathbb{R}\setminus[-1,1]} |x|^s f(x)dx.$$

This implies that the residue at $s = 1$ of $\beta^*(u_{\Gamma_p}^P)^s$ is a density supported at the exceptional divisor (which is given in these coordinates by $y_1^0 = 0$), and integrating this density against the constant function $\underline{1}_Y$ gives what is called the residue of the graph Γ_p,

$$\operatorname{res}_P \Gamma_p = \operatorname{res}_{s=1} \beta^*(u^P_{\Gamma_p})^s[\underline{1}_Y] = -\frac{2}{d_{\Gamma_p}} \int_{\mathcal{E}} f_{\Gamma_p}.$$

(The exceptional divisor can actually be oriented in such a way that f_{Γ_p} is a degree $(d_{\Gamma_p} - 1)$ differential form.)

Let us now come back to the case of Γ_3 which is not primitive but has a nested set of three divergent subgraphs. Raising (13) to a power s results in a pole at $s = 1$ of order 3. The Laurent coefficient a_{-3} of $(s-1)^{-3}$ is supported on

$$\mathcal{E}_{1234} \cap \mathcal{E}_{234} \cap \mathcal{E}_{34},$$

for this is the set given in local coordinates by $y^0_1 = y^0_2 = y^0_3 = 0$. Similarly, the coefficient of $(s-1)^{-2}$ is supported on

$$(\mathcal{E}_{1234} \cap \mathcal{E}_{234}) \cup (\mathcal{E}_{1234} \cap \mathcal{E}_{34}) \cup (\mathcal{E}_{234} \cap \mathcal{E}_{34})$$

and the coefficient of $(s-1)^{-1}$ on

$$\mathcal{E}_{1234} \cup \mathcal{E}_{234} \cup \mathcal{E}_{34}.$$

(The non-negative part of the Laurent series is supported everywhere on Y.) Write $|dy| = |dy^0_1 \dots dy^3_3|$. In order to compute the coefficient a_{-3}, one needs to integrate f_{Γ_3}, restricted to the subspace $y^0_1 = y^0_2 = y^0_3 = 0$:

$$f_{\Gamma_3} = \frac{|dy|}{(1+\underline{y}_1^2)(1+\underline{y}_2^2)(1+\underline{y}_3^2)} \cdot \frac{1}{((1+y^0_2)^2 + (\underline{y}_1 + y^0_2\underline{y}_2)^2)((1+y^0_3)^2 + (\underline{y}_2 + y^0_3\underline{y}_3)^2)},$$

where $\underline{y}_i$ denotes the 3-vector (y^1_i, y^2_i, y^3_i). Consequently

$$f_{\Gamma_3}|_{y^0_1=y^0_2=y^0_3=0} = \frac{|dy|}{(1+\underline{y}_1^2)^2,(1+\underline{y}_2^2)^2(1+\underline{y}_3^2)^2} = f_{\Gamma_1}^{\otimes 3}$$

where Γ_1 is the primitive graph with two vertices and two parallel edges joining them:

$\Gamma_1 =$ 

. (14)

The chart where (13) holds covers actually all of Y_P up to a set of measure zero where there are no additional divergences. It suffices therefore to integrate in these coordinates only. Several charts must be taken into account however when there is more than one maximal nested set. In conclusion,

$$a_{-3}[\underline{1}_Y] = (\mathrm{res}_P\, \Gamma_1)^3, \tag{15}$$

which is a special case of a theorem in [10] relating pole coefficients of $\beta^*(u^P_\Gamma)^s$ to residues of graphs obtained from Γ by contraction of divergent subgraphs.

But the ultimate reason to introduce the resolution of singularities in the first place is as follows: In order to obtain an extension (renormalization) of u^P_Γ, one can now simply remove the simple pole at $s = 1$ along each component of the exceptional divisor:

$$w^P_{\Gamma_3} = \frac{f_{\Gamma_3}}{|y^0_1 y^0_2 y^0_3|}, \quad (w^P_{\Gamma_3})_R = \frac{f_{\Gamma_3}}{|y^0_1|_{\mathrm{fin}}|y^0_2|_{\mathrm{fin}}|y^0_3|_{\mathrm{fin}}}.$$

The second distribution $(w^P_{\Gamma_3})_R$ is defined on all of Y, and consequently $\beta_*(w^P_{\Gamma_3})_R$ on all of A_P. It agrees with $u^P_{\Gamma_3}$ on test functions having support in A°_P and is therefore an extension. The difference between $w^P_{\Gamma_3}$ and $(w^P_{\Gamma_3})_R$ is a distribution supported on the exceptional divisor which gives rise to a candidate for a counterterm in the Lagrangian.

I call this renormalization scheme *local minimal subtraction*, because locally, along each component of the exceptional divisor, the simple pole is removed in a "minimal way", changing only the principal part of the Laurent series. See [10] for a proof that this results in local counterterms, a necessary condition for the extension to be a physically consistent one.

3.3 Momentum space. In momentum space, the bad definition of the position space Feynman distribution at certain diagonals $\bigcap D_e$ is translated by a Fourier transform into ill-defined (divergent) integrals with divergences at certain strata at infinity. For example, the position space integral $(\mathcal{M}, u^P_{\Gamma_1} = u^2_{0,P})$ in $d = 4$ dimensions for the graph Γ_1 (see (14)) has a divergence at 0 (which is the image πD_{12} of the diagonal). A formal Fourier transform would turn the pointwise product $u^2_{0,P}$ into a convolution product

$$(\mathcal{F} u^2_{0,P})(P) = \int u_{0,M}(p) u_{0,M}(p-P) d^4 p.$$

In fact the right hand side is exactly $U^M_{\Gamma_1}(P)[\underline{1}_{A_{\Gamma_1}}]$, in agreement with Proposition 2.1. It does not converge at ∞. (In order to see this we actually only need $U^M_{\Gamma_1}|_{P=0} = u^M_{\Gamma_1}$, not the dependence upon external momenta.)

On the other hand, the infrared singularities are to be found at affine subspaces in momentum space. Of course the program sketched in the previous section can be applied to the momentum space Feynman distribution as well. A resolution of singularities for the relevant strata at infinity can be found, and the pullback of the momentum space Feynman distribution can be extended onto all the irreducible components of the exceptional divisor. But I want to use this section in order to sketch another, algebraic, approach to the momentum space renormalization problem, which is due to Connes and Kreimer [53], [29], [30].

Assume $U_\Gamma^M[\underline{1}_{A_M}]$ varies holomorphically with d in a punctured disk around $d = 4$. Physicists call this dimensional regularization [39], [32]: any integral of the form $\int d^4 p u(p) dp$ is replaced by a d-dimensional integral $\int d^d p u(p) dp$. In this way we can consider U_Γ^M as a distribution on *all of* $A_P \times A_M$ with values in $\mathcal{R} = \mathbb{C}[[(d-4)^{-1}, (d-4)]]$, the field of Laurent series in $d-4$. If $U_\Gamma^M[f]$ is not convergent in $d = 4$ dimensions, then there will be a pole at $d = 4$.

Now let $\sigma_\Gamma \in \mathcal{D}'(A_P)$ be a distribution with compact support. Since the distribution U_Γ^M is smooth in the P_v, we can actually integrate it against the distribution σ_Γ. (For example, if $\sigma_\Gamma = \delta_0(|P_{v_1}|^2 - E_1) \otimes \cdots \otimes \delta_0(|P_{v_n}^2| - E_n)$ then this amounts simply to evaluating U_Γ^M on the subspaces $|P_{v_1}|^2 = E_1, \ldots, |P_{v_2}|^2 = E_n$.) In any case we have a map

$$\phi\colon (\Gamma, \sigma_\Gamma) \mapsto U_\Gamma^M[\underline{1}_{A_M} \otimes \sigma_\Gamma] \in \mathcal{R}$$

sending pairs to Laurent series. Now let $\mathcal{H}$ be the polynomial algebra over $\mathbb{C}$ generated by isomorphism classes of connected core divergent graphs Γ of a given renormalizable quantum field theory. Define a coproduct Δ by

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\substack{\gamma_1 \sqcup \cdots \sqcup \gamma_k \subsetneq \Gamma \\ \text{conn. core div.}}} \gamma_1 \ldots \gamma_k \otimes \Gamma /\!/ (\gamma_1 \sqcup \cdots \sqcup \gamma_k).$$

The notation $\Gamma/\!/\gamma$ means that any connected component of γ inside Γ is contracted to a (separate) vertex. By standard constructions [29], $\mathcal{H}$ becomes a Hopf algebra, called the Connes–Kreimer Hopf algebra. Denote the antipode by S. Now let $\mathcal{H}_\sigma$ be the corresponding Hopf algebra of pairs (Γ, σ_Γ). (In order to define this Hopf algebra of pairs, one needs the extra condition that σ_Γ vanishes on all vertices that have no external edges, a standard assumption if one considers only graphs of a fixed renormalizable theory.)

The map $\phi\colon \mathcal{H}_\sigma \to \mathcal{R}$ is a homomorphism of unital $\mathbb{C}$-algebras. The space of these maps $\mathcal{H}_\sigma \to \mathcal{R}$ is a group with the convolution product

$$\phi_1 \star \phi_2 = m(\phi_1 \otimes \phi_2)\Delta.$$

On $\mathcal{R}$, there is the linear projection

$$R\colon (d-4)^n \mapsto \begin{cases} 0 & \text{if } n \geq 0, \\ (d-4)^n & \text{if } n < 0 \end{cases} \tag{16}$$

onto the principal part.

Theorem 3.1 (Connes, Kreimer). *The renormalized Feynman integral* $\phi_R(\Gamma, \sigma_\Gamma)|_{d=4}$ *and the counterterm* $S_R^\phi(\Gamma, \sigma_\Gamma)$ *are given as follows. I write* $\underline{\Gamma}$ *for the pair* (Γ, σ_Γ):

$$S_R^\phi(\underline{\Gamma}) = -R(\phi(\underline{\Gamma}) + \sum_{\substack{\gamma=\gamma_1\sqcup\cdots\sqcup\gamma_k\subsetneq\Gamma \\ \text{conn. core div.}}} S_R^\phi(\underline{\gamma})\phi(\underline{\Gamma}/\!/\underline{\gamma})),$$

$$\phi_R(\underline{\Gamma}) = (1-R)(\phi(\underline{\Gamma}) + \sum_{\substack{\gamma=\gamma_1\sqcup\cdots\sqcup\gamma_k\subsetneq\Gamma \\ \text{conn. core div.}}} S_R^\phi(\underline{\gamma})\phi(\underline{\Gamma}/\!/\underline{\gamma})).$$

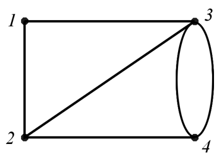
These expressions are assembled from the formula for the antipode and the convolution product. Combinatorially, the Hopf algebra encodes the BPHZ recursion [46] and Zimmermann's forest formula [90]. The theorem can be interpreted as a Birkhoff decomposition of the character ϕ into $\phi_- = S_R^\phi$ and $\phi_+ = \phi_R$ [30].

The renormalization scheme described here is what I call *global minimal subtraction*, because in the target field $\mathcal{R}$, when all local information has been integrated out, the map $1 - R$ removes only the entire principal part at $d = 4$. This coincides with the renormalization scheme described in [28].

In the case of $m = 0$ and zero-momentum transfer (all but two external momenta set to 0) one knows that at $d = 4$,

$$\phi_R(\Gamma) = \sum_{n=0}^{N} p_n(\Gamma)(\log |P|^2/\mu^2)^n, \quad p_n(\Gamma) \in \mathbb{R}, \tag{17}$$

where μ is an energy scale, and the σ_Γ can be dropped for convenience. Let us now do our standard example

$\Gamma_3 =$ 

using the Hopf algebra. We interpret Γ_3 as a graph in ϕ^4 theory, so we think of two external edges at the first vertex, one at the second, and one at the fourth. Recall the momentum space Feynman rules (3) for Γ_3. Let $P_2 = 0$ and write $P = P_1 = -P_4$ such that P_1 is the sum of the two external momenta entering at the first vertex. Then

$$\phi(\Gamma_3) = \int \frac{d^d p_1 d^d p_2 d^d p_3}{p_1^2(p_1+P)^2 p_2^2(p_1+p_2+P)^2 p_3^2(p_2+p_3-P)^2} \in \mathcal{R}.$$

This integral can be evaluated as a Laurent series in $d = 4$ using standard techniques [28]. It has a pole of order 3 at $d = 4$, and one might think of simply taking $(1-R)\phi(\Gamma_3)$ as a renormalized value, for this kills the principal part, and the limit at $d = 4$ may be taken. But the resulting counterterms would not be local ones, and the renormalization would be physically inconsistent. The benefit of the Hopf algebra approach is that the necessary correction terms are provided right away.

Let again γ_1 be the full subgraph with vertices 3 and 4, and γ_2 the full subgraph with vertices 2, 3 and 4. Then

$$\phi_R(\Gamma_3) = (1-R)(\phi(\Gamma_3)-(R\phi(\gamma_2))\phi(\Gamma_3/\!/\gamma_2)+R((R\phi(\gamma_1))\phi(\gamma_2/\!/\gamma_1))\phi(\Gamma_3/\!/\gamma_2)).$$

Observe that, as a coincidental property of our example, $\Gamma_3/\!/\gamma_2 \cong \gamma_2/\!/\gamma_1 \cong \gamma_1$ (compare this with (15), (21)).

The Hopf algebra approach to renormalization has brought up a number of surprising connections to other fields; see for example [30], [31], [37], [68], [41], [79], [64], [80], [59]. Other developments starting from the Connes–Kreimer theory can be found in [32]. Kreimer and van Suijlekom have shown that gauge and other symmetries are compatible with the Hopf algebra structure [55], [85], [86], [84], [61].

A sketch of how the combinatorics of the Hopf algebra relates to the resolution of singularities in the previous section and to position space renormalization can be found in [10]; see also Section 3.6.

3.4 Parametric representation. In the parametric representation introduced in Section 2.2, the divergences can be found at certain intersections of the coordinate hyperplanes $A_e = \{a_e = 0\}$. This is in fact one of the very reasons why the parametric representation was introduced: Consider for example the divergent integral $(\mathbb{R}^4, u_{0,M}^2)$, with $u_{0,M} = \frac{1}{|p|^2}$,

$$\int \frac{d^4p}{|p|^4} = \int\int_0^\infty\int_0^\infty \exp(-a_1|p|^2 - a_2|p|^2)\, da_1\, da_2\, d^4p$$

in the sense of Definition 2.1. (In this section, instead of (A, u) I will simply write $\int_A u(x)dx$.) The integral on the left-hand side is divergent both at 0 and at ∞. But splitting it into the two parts at the right, and interchanging the d^4p with the da_1da_2 integrations leaves a Gaussian integral

$$\int \exp(-\tfrac{c}{2}|p|^2)d^dp = (2\pi/c)^{d/2},$$

which is convergent but at the expense of getting $(a_1 + a_2)^2$ in the denominator: The integral

$$\int_0^\infty\int_0^\infty \frac{da_1da_2}{(a_1+a_2)^2}$$

has a logarithmic singularity at 0 and at ∞. This can be seen by blowing up the origin in $\mathbb{R}^2_{\geq 0}$, and pulling back:

$$\int_0^\infty\int_0^\infty \frac{db_1db_2}{b_1(1+b_2)^2}.$$

In other words, the trick with the parametric parameterization (called the *Schwinger trick* in [15]), does not get rid of any divergences. It just moves them into another, lower-dimensional space.

Again, it is useful to have a resolution of singularities in order to separate the various singularities and divergences of a graph along irreducible components of a divisor with normal crossings. The most obvious and efficient such resolution is given in [15], [16].

Let Γ be core. For a subgraph $E(\gamma) \subseteq E(\Gamma)$, let

$$L_\gamma = \bigcap_{e \in E(\gamma)} A_e = \{a_e = 0,\, e \in E(\gamma)\},$$

a linear subspace. Set $\mathcal{L}_{\text{core}} = \{L_\gamma : \gamma \text{ is a core subgraph of } \Gamma\}$, and

$$\mathcal{L}_0 = \{\text{minimal element of } \mathcal{L}_{\text{core}}\} = \{0\},$$
$$\mathcal{L}_{n+1} = \{\text{minimal elements of } \mathcal{L}_{\text{core}} \setminus \textstyle\bigsqcup_{i=0}^n \mathcal{L}_i\}.$$

This partition of $\mathcal{L}_{\text{core}}$ is made in such a way that (see [16], Proposition 3.1) a sequence of blowups

$$\gamma\colon Z_S \to \cdots \to A_S \tag{18}$$

is possible which starts by blowing up $\mathcal{L}_0$ and then successively the strict transforms of the elements of $\mathcal{L}_1, \mathcal{L}_2, \ldots$. This ends up with Z_S a manifold with corners. The map γ is of course defined not only as a map onto $A_S = \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$ but as a birational map $\gamma\colon \mathcal{Z}_S \to \mathbb{C}^{|E(\Gamma)|}$, with $\mathcal{Z}_S$ a smooth complex variety. The total exceptional divisor $\mathcal{E}$ has normal crossings, and one component $\mathcal{E}_L$ for each $L \in \mathcal{L}_{\text{core}}$. (In the language of Section 3.2, $\mathcal{L}_{\text{core}}$ is the "building set".) Moreover,

$$\mathcal{E}_{L_1} \cap \cdots \cap \mathcal{E}_{L_k} \neq 0 \iff \text{the } L_i \text{ are totally ordered by inclusion.}$$

Since the coordinate divisor $\{a_e = 0 \text{ for some } e \in E(\Gamma)\}$ already has normal crossings by definition, the purpose of these blowups is really only to pull out into codimension 1 all the intersections where there are possibly singularities or divergences, and to separate the integrable singularities of the integrand from this set as much as possible.

Note that in the parametric situation where the domain of integration is the manifold with corners $\mathbb{R}_{\geq 0}^{|E(\Gamma)|}$, the blowups do not introduce an orientation issue on the real locus.

For the example graph Γ_3 of the previous sections (see (12)),

$$u_{\Gamma_3}^S = \frac{da_1 \ldots da_6}{((a_1 + a_2)((a_3 + a_4)(a_5 + a_6) + a_5 a_6) + a_3 a_4 a_5 + a_3 a_4 a_6 + a_3 a_5 a_6)^{d/2}}$$

we examine the pullback of $u_{\Gamma_3}^S$ onto Z_S. There are various core subgraphs to consider, but it is easily seen, in complete analogy with (11), that the divergences are located only at L_{Γ_3}, L_{γ_2} and L_{γ_1}, where γ_1 is the full subgraph with vertices 3 and 4, and γ_2 the full subgraph with vertices 2, 3 and 4. In order to see the divergences in Z_S, it therefore suffices to look in a chart where $\mathcal{E}_{L_{\Gamma_3}}$, $\mathcal{E}_{L_{\gamma_2}}$ and $\mathcal{E}_{L_{\gamma_1}}$ intersect. In such a chart, given by coordinates $b_1 = a_1$, $b_2 = a_2/a_1$, $b_3 = a_3/a_1$, $b_4 = a_4/a_1$, $b_5 = a_5/a_3$, $b_6 = a_6/a_5$, we have

$$\gamma^* u_{\Gamma_3}^S = \frac{db_1 \ldots db_6}{b_1 b_3 b_5 ((1 + b_2)((1 + b_6)(1 + b_4) + b_5 b_6) + b_3(b_5 b_6 + b_4 b_6 + b_4))^{d/2}} \tag{19}$$

Now we are in a very similar position as that in the previous section. If Γ_p is a primitive graph, then there is only the origin $0 \in A_S$ which needs to be blown up in order to isolate the divergence. Since $u^S_{\Gamma_3}$ depends explicitly on d in the exponent, let us use d as an analytic regulator. One finds, using for example coordinates $b_1 = a_1$, $b_i = a_i/a_1$, $i \neq 1$, in a neighborhood of $d = 4$,

$$\gamma^* u^S_{\Gamma_p}(d) = \left(\frac{\delta_0(b_1)}{d-4} + \text{finite}\right) g_{\Gamma_p}$$

with $g_{\Gamma_p} \in L^1_{\text{loc}}$. (If one wants moreover a regular g_{Γ_p} one needs to perform the remaining blowups in (18).) Then we define

$$\operatorname{res}_S \Gamma_p = (\operatorname{res}_{d=4} \gamma^* u^S_{\Gamma_p}(d))[\underline{1}] = \int_{b_1=0, b_i \geq 0} g_{\Gamma_p} = \int_\sigma \frac{\Omega}{\Psi^2_{\Gamma_p}}, \tag{20}$$

where $\sigma = \{a_i \geq 0\} \subset \mathbb{P}^{|E(\Gamma)|-1}(\mathbb{R})$ and $\Omega = \sum_{n=1}^{|E(\Gamma)|} (-1)^n a_n da_1 \wedge \cdots \wedge \widehat{da_n} \wedge \cdots \wedge da_{|E(\Gamma)|}$. The last integral on the right is a projective integral, meaning that the a_i are interpreted as homogeneous coordinates of $\mathbb{P}^{|E(\Gamma)|-1}$. By choosing affine coordinates b_i, one finds that it is identical with the integral of g_{Γ_p} over the exceptional divisor intersected with the total inverse image of A_S. Coming back to the non-primitive graph Γ_3 (see (19)) we find, in complete analogy with Section 3.2, that

$$u^S_{\Gamma_3}(d) = \sum_{n \geq -3}^{\infty} c_n (d-4)^n$$

in a neighborhood of $d = 4$, and

$$c_{-3}[\underline{1}_{A_S}] = (\operatorname{res}_S \Gamma_1)^3, \tag{21}$$

which is easily seen by sending b_1, b_3, b_5 to 0 in (19): $g_{\Gamma_3}|_{b_1=b_3=b_5=0} = g^{\otimes 3}_{\Gamma_1}$.

Similarly, one can translate the results of Section 3.2 and [10] into this setting and obtain a renormalization (extension of u^S_Γ) by removing the simple pole along each component of the irreducible divisor. In Section 4.5 a different, motivic renormalization scheme for the parametric representation will be studied, following [16].

3.5 Dyson–Schwinger equations. Up to now we have only considered single Feynman graphs, with internal edges interpreted as virtual particles, and parameters such as the mass subjected to renormalization. Another approach is to start with the full physical particles from the beginning, that is, with the non-perturbative objects. Implicit equations satisfied by the physical particles (full propagators) and the physical interactions (full vertices) are called *Dyson–Schwinger equations.* The equations can be imposed in a Hopf algebra of Feynman graphs [23], [58], [11], [57], [89] and turn into systems of integral equations when Feynman rules are applied.

For general configurations of external momenta, Dyson–Schwinger equations are extremely hard to solve. But if one sets all but two external momenta to 0, a situation called zero-momentum transfer (see (17)), then the problem simplifies considerably.

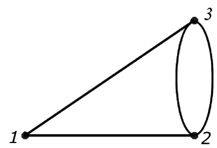
In [60], an example of a linear Dyson–Schwinger equation is given which can be solved nonperturbatively by a very simple Ansatz. More difficult non-linear Dyson–Schwinger equations, and finally systems of Dyson–Schwinger equations as above, are studied in [62], [63], [82], [83]; see also [89], [56], [40].

3.6 Remarks on minimal subtraction. I come back at this point to the difference between what I call *local* (Section 3.2) and *global* (Section 3.3) *minimal subtraction*, which, I think, is an important one.

I tried to emphasize in the exposition of the previous sections that the key concepts of renormalization are largely independent of whether momentum space, position space, or parameter space Feynman rules are used. This is immediately seen in the Connes–Kreimer Hopf algebra framework where a graph Γ and some external information σ_Γ are sent directly to a Laurent series in $d-4$. For this we do not get to see and do not need to know if the integral has been computed in momentum, position, or parameter space. They all produce the same number (or rather Laurent series), provided the same regularization is chosen for all three of them.

In position space, where people traditionally like to work with distributions as long as possible and integrate them against a test function only at the very end (or even against the constant function $\underline{1}$, the adiabatic limit), one is tempted to define the Feynman rules as a map into a space of distribution-valued Laurent series, as we have done in [10]. But one has to be aware that this space of distribution-valued Laurent series does not necessarily qualify as a replacement for the ring $\mathcal{R}$ in Section 3.3 if one looks for a new Birkhoff decomposition. In general, many questions and misconceptions that I have encountered in this area can be traced back to deciding at which moment one integrates, and minimal subtraction seems to be a good example of this.

Let me now give a detailed comparison of what happens in local and global minimal subtraction, respectively. Assume for example the massless graph in 4 dimensions,

$\Gamma =$  .

Clearly Γ itself and the full subgraph γ on the vertices 2 and 3 are logarithmically divergent. No matter which kind of Feynman rules we use, assume there is a regularized Feynman distribution $u_\Gamma(\epsilon)$ varying holomorphically in a punctured disk around $\epsilon = 0$, with a finite order pole at $\epsilon = 0$. Assume after resolution of singularities that the regularized Feynman distribution, pulled back onto the smooth model, has a simple pole supported on the component $\mathcal{E}_\Gamma$ of the total exceptional divisor (for the superficial divergence), and another on the component $\mathcal{E}_\gamma$ (for the subdivergence).

Let $\mathcal{E}_\Gamma = \{y_\Gamma = 0\}$ and $\mathcal{E}_\gamma = \{y_\gamma = 0\}$ in local coordinates $y_\Gamma, y_\gamma, y_3, \dots, y_n$. We have

$$u_\Gamma(\epsilon) = \left(\frac{\delta_0(y_\Gamma)}{\epsilon} + |y_\Gamma|_{\text{fin}}(\epsilon)\right)\left(\frac{\delta_0(y_\gamma)}{\epsilon} + |y_\gamma|_{\text{fin}}(\epsilon)\right) f_\Gamma(\epsilon), \tag{22}$$

where f_Γ is locally integrable and smooth in y_Γ and y_γ, such that in particular $f_\Gamma(\epsilon)$ is holomorphic in ϵ. There is accordingly a second order pole supported at $\mathcal{E}_\Gamma \cap \mathcal{E}_\gamma$. We know from [10], as was also sketched in Section 3.2, that the leading coefficient of this second order pole is a product of delta functions restricting it to $\mathcal{E}_\Gamma \cap \mathcal{E}_\gamma$ times the residue of γ times the residue of $\Gamma /\!/ \gamma$.

Consequently, integrating $u_\Gamma(\epsilon)$ against a fixed function χ (for a first reading take $\chi = \underline{1}$, but in the massless case one has to worry about infared divergences) provides a Laurent series

$$u_\Gamma(\epsilon)[\chi] = a_{-2}\epsilon^{-2} + a_{-1}\epsilon^{-1} + a_0\epsilon^0 + \cdots .$$

Since γ and $\Gamma /\!/ \gamma$ are primitive,

$$u_\gamma(\epsilon)[\chi] = b_{-1}\epsilon^{-1} + b_0\epsilon^0 + b_1\epsilon^1 + \cdots ,$$
$$u_{\Gamma/\!/\gamma}(\epsilon)[\chi] = c_{-1}\epsilon^{-1} + c_0\epsilon^0 + b_1\epsilon^1 + \cdots .$$

We know from the previous remarks that $a_{-2} = \operatorname{res}(\gamma)\operatorname{res}(\Gamma/\!/\gamma) = b_{-1}c_{-1}$ and similarly $a_{-1} = b_{-1}c_0 + g$, where I do not want to specify g.

Let me now compare local and global minimal subtraction in this example. Local minimal subtraction is defined on distribution-valued Laurent series, but global minimal subtraction only on $\mathbb{C}$-valued Laurent series. Therefore we need to integrate everything out before comparing. I start with local minimal subtraction (LMS). In order to get from (22) to

$$(u_\Gamma)_{R,\text{LMS}}(\epsilon) = |y_\Gamma|_{\text{fin}}(\epsilon)|y_\gamma|_{\text{fin}}(\epsilon) f_\Gamma(\epsilon)$$

one has to subtract three terms from (22):

$$R^\Gamma_{\text{LMS}} u_\Gamma(\epsilon) = \frac{\delta_0(y_\Gamma)}{\epsilon}\left(\frac{\delta_0(y_\gamma)}{\epsilon} + |y_\gamma|_{\text{fin}}(\epsilon)\right) f_\Gamma(\epsilon),$$
$$R^{\gamma,\Gamma/\!/\gamma}_{\text{LMS}} u_\Gamma(\epsilon) = \left(\frac{\delta_0(y_\Gamma)}{\epsilon} + |y_\Gamma|_{\text{fin}}(\epsilon)\right)\frac{\delta_0(y_\gamma)}{\epsilon} f_\Gamma(\epsilon),$$
$$-RR^{\gamma,\Gamma/\!/\gamma}_{\text{LMS}} u_\Gamma(\epsilon) = -\frac{\delta_0(y_\Gamma)}{\epsilon}\frac{\delta_0(y_\gamma)}{\epsilon} f_\Gamma(\epsilon).$$

The first term eliminates the pole supported on $\mathcal{E}_\Gamma$, such that $u_\Gamma - R^\Gamma_{\text{LMS}} u_\Gamma$ has only a simple pole supported on $\mathcal{E}_\gamma$ left. On the other hand, $u_\Gamma - R^{\gamma,\Gamma/\!/\gamma}_{\text{LMS}} u_\Gamma$ has only a simple pole supported on $\mathcal{E}_\Gamma$ left, and the third term is a correction term supported on $\mathcal{E}_\gamma \cap \mathcal{E}_\Gamma$ accounting for what has been subtracted twice. In summary,

$$(u_\Gamma)_{R,\text{LMS}}(\epsilon) = u_\Gamma(\epsilon) - R^\Gamma_{\text{LMS}} u_\Gamma(\epsilon) - R^{\gamma,\Gamma/\!/\gamma}_{\text{LMS}} u_\Gamma(\epsilon) + RR^{\gamma,\Gamma/\!/\gamma}_{\text{LMS}} u_\Gamma(\epsilon) \tag{23}$$

is the result of local minimal subtraction.

Let us now integrate out (23).

$$\begin{aligned} u_\Gamma(\epsilon)[\chi] &= a_{-2}\epsilon^{-2} + a_{-1}\epsilon^{-1} + a_0\epsilon^0 + \cdots, \\ R^\Gamma_{\mathrm{LMS}} u_\Gamma(\epsilon)[\chi] &= a_{-2}\epsilon^{-2} + g\epsilon^{-1} + h\epsilon^0 + \cdots, \\ R^{\gamma,\Gamma/\!/\gamma}_{\mathrm{LMS}} u_\Gamma(\epsilon)[\chi] &= a_{-2}\epsilon^{-2} + b_{-1}c_0\epsilon^{-1} + b_{-1}c_1\epsilon^0 + \cdots, \\ -RR^{\gamma,\Gamma/\!/\gamma}_{\mathrm{LMS}} u_\Gamma(\epsilon)[\chi] &= a_{-2}\epsilon^{-2}, \end{aligned}$$

These equations follow from (22), and I do not want to specify h. Consequently

$$(u_\Gamma)_{R,\mathrm{LMS}}(\epsilon)[\chi] = a_0 - b_{-1}c_1 - h \text{ as } \epsilon \to 0.$$

In global minimal subtraction (GMS), where $R_{\mathrm{GMS}} = R$ as in (16), something different happens.

$$\begin{aligned} R_{\mathrm{GMS}}(u_\Gamma(\epsilon)[\chi]) &= a_{-2}\epsilon^{-2} + a_{-1}\epsilon^{-1}, \\ (R_{\mathrm{GMS}} u_\gamma(\epsilon)[\chi]) u_{\Gamma/\!/\gamma}(\epsilon)[\chi] &= b_{-1}c_{-1}\epsilon^{-2} + b_{-1}c_0\epsilon^{-1} + b_{-1}c_1\epsilon^0 + \cdots, \\ -R_{\mathrm{GMS}}(R_{\mathrm{GMS}} u_\gamma(\epsilon)[\chi]) u_{\Gamma/\!/\gamma}(\epsilon)[\chi]) &= b_{-1}c_{-1}\epsilon^{-2} + b_{-1}c_0\epsilon^{-1}. \end{aligned}$$

The first subtraction $u_\Gamma[\chi] - R_{\mathrm{GMS}}(u_\Gamma[\chi])$ removes the poles everywhere, also the one supported on $\mathcal{E}_\gamma$ which has nothing to do with the superficial divergence. The third and fourth term restore the locality of counterterms. We have

$$(u_\Gamma)_{R,\mathrm{GMS}}(\epsilon)[\chi] = a_0 - b_{-1}c_1 \quad \text{as } \epsilon \to 0.$$

In summary: Unless $h = 0$, local and global minimal subtraction differ by a finite renormalization. Moreover, although there is a one-to-one-correspondence between terms to be subtracted in LMS and GMS, the values of those single terms do not agree. It seems to me that GMS is a quite clever but somehow special trick of defining the subtraction operator R on $\mathbb{C}$-valued Laurent series where all the geometric information (i.e., where the pole is supported) has been forgotten.

In [10] it is shown how to relate, for a general graph Γ, the combinatorics of the total exceptional divisor of the resolution of singularities to the Connes–Kreimer Hopf algebra of Feynman graphs, such that the example presented here is a special case of a more general result. A similar analysis applies to other local renormalization prescriptions, called subtraction at fixed conditions in [10], as well.

4 Motives and residues of Feynman graphs

4.1 Motives, Hodge realization and periods. Much of the present interest in Feynman integrals is due to the more or less obvious fact that there is something *motivic* about them. In order to understand and appreciate this, one obviously needs to have an idea of what a motive is. I am not an expert in this area and will not even attempt to provide much background to the notion of motive. See [4] for an often cited introduction to the subject, which I follow closely in the beginning of this section.

The theory of motives is a means to unify the various cohomology theories known for algebraic varieties X over a number field k. Such cohomology theories include the algebraic de Rham and the Betti cohomology, but there are many others. The algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X)$ is defined over the ground field k, and Betti cohomology $H^\bullet_{\mathrm{B}}(X;\mathbb{Q})$ is the singular cohomology of $X(\mathbb{C})$ with rational coefficients.

A motive of a variety is supposed to be a piece of a universal cohomology, such that all the usual cohomology theories (functors from varieties to graded vector spaces) factor through the category of motives. A particular cohomology theory is then called a *realization*. For example, the combination of de Rham and Betti cohomology, giving rise to a Hodge structure, is called the *Hodge realization*.

The theory of motives is not yet complete. Only for the simplest kind of algebraic varieties, smooth projective ones, has a category of motives with the desired properties been constructed. These motives are called *pure*. For general, i.e., singular or non-projective varieties, the theory is conjectural in the sense that only a triangulated category as a candidate for the derived category of the category of these motives, called *mixed motives*, exists.

Let X be a smooth variety over $\mathbb{Q}$. Let $H^\bullet_{\mathrm{dR}}(X)$ denote the algebraic de Rham cohomology of X, a graded $\mathbb{Q}$-vector space, and $H^\bullet_{\mathrm{B}}(X;\mathbb{Q})$ the rational Betti cohomology (singular cohomology of the complex manifold $X(\mathbb{C})$ with rational coefficients), a graded $\mathbb{Q}$-vector space. A *period* of X is by definition a matrix element of the *comparison isomorphism* (integration)

$$H^\bullet_{\mathrm{dR}}(X)\otimes_{\mathbb{Q}}\mathbb{C}\cong H^\bullet_{B}(X;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$$

for a suitable choice of basis. A period is therefore in particular an integral of an algebraic differential form over a topological cycle on $X(\mathbb{C})$. A standard example is the case of an elliptic curve X defined by the equation $y^2 = x(x-1)(x-\lambda)$, $\lambda\in\mathbb{Q}\setminus\{0,1\}$. A basis element of $H^1_{\mathrm{dR}}(X)$ is the 1-form $\omega=\frac{dx}{2y}$ and a basis of the singular cohomology $H^1_{\mathrm{B}}(X)$ is given by the duals of two circles around the cut between 0 and 1, resp. the cut between 1 and ∞. Integrating ω against these cycles gives the generators of the period lattice of X.

Similarly, matrix elements of a comparison isomorphism between *relative* cohomologies of pairs (X,A) are called *relative periods*. Many examples considered below will be relative periods.

4.2 Multiple zeta values, mixed Tate motives and the work of Belkale and Brosnan.

Let Γ be a primitive Feynman graph. I assume $d=4$ and $m=0$. Recall the graph polynomial

$$\Psi_\Gamma=\sum_{T\,\mathrm{st\ of}\,\Gamma}\prod_{e\notin E(T)}a_e\in\mathbb{Z}[a_e : e\in E(\Gamma)]$$

from (5). The sum is over the spanning trees of Γ. Following [15], we have a closer look at the parametric residue

$$\mathrm{res}_S\,\Gamma = \int_\sigma \frac{\Omega}{\Psi_\Gamma^2}$$

introduced in (20).

Let $X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{P}^{|E(\Gamma)|-1}$ and $CX_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{A}^{|E(\Gamma)|}$ its affine cone. X_Γ resp. CX_Γ are called the *projective* resp. *affine graph hypersurface*. The chain of integration is $\sigma = \{a_e \geq 0\} \subseteq \mathbb{P}^{|E(\Gamma)|-1}(\mathbb{R})$, and $\Omega = \sum(-1)^n a_n da_1 \wedge \cdots \wedge \widehat{da_n} \wedge \cdots \wedge da_{|E(\Gamma)|}$.

The residue $\mathrm{res}_S\,\Gamma$ already looks like a relative period, since σ has its boundary contained in the coordinate divisor $\Delta = \bigcup_{e\in E(\Gamma)}\{a_e = 0\}$, and the differential form $\frac{\Omega}{\Psi_\Gamma^2}$ is algebraic (i.e., regular) in $\mathbb{P}^{|E(\Gamma)|-1} \setminus X_\Gamma$. But in general $X_\Gamma \cap \Delta$ is quite big, and $\frac{\Omega}{\Psi_\Gamma^2} \notin H_{\mathrm{dR}}^{|E(\Gamma)|-1}(\mathbb{P}^{|E(\Gamma)|-1} \setminus X_\Gamma, \Delta \setminus (X_\Gamma \cap \Delta))$.

The solution is of course to work in the blowup $\mathcal{Z}_S$ of Section 3.4 where things are separated. Let $\mathcal{P}_S$ be the variety obtained from $\mathbb{P}^{|E(\Gamma)|-1}$ by regarding all elements of the $\mathcal{L}_n$ $(n \geq 1)$ in Section 3.4 as subspaces of $\mathbb{P}^{|E(\Gamma)|-1}$ and starting the blowup sequence at $n = 1$ instead of $n = 0$.

In [15], [16] it is shown that $\mathcal{P}_S$ has the desired properties: the strict transform of X_Γ does not meet the strict transform of σ. In this way $\mathrm{res}_S\,\Gamma$ is a relative period of the pair

$$(\mathcal{P}_S \setminus Y_\Gamma, B \setminus (B \cap Y_\Gamma)),$$

where Y_Γ is the strict transform of X_Γ and B the total transform of the coordinate divisor Δ.

We call $\mathrm{res}_S\,\Gamma$ a *Feynman period* of Γ.

An empirical observation due to Broadhurst and Kreimer [21], [22] was that all Feynman periods computed so far are rational linear combinations of multiple zeta values.

A *multiple zeta value* of depth k and weight $s = s_1 + \cdots + s_k$ is a real number defined as

$$\zeta(s_1, \ldots, s_k) = \sum_{1\leq n_k<\cdots<n_1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}},$$

where $s_1 \geq 2$ and $s_2, \ldots, s_k \geq 1$. For $k = 1$ one obtains the values of the Riemann zeta function at integer arguments ≥ 2, whence the name.

By an observation due to Euler and Kontsevich, multiple zeta values can be written as iterated integrals

$$\zeta(s_1, \ldots, s_k) = \int_{0<t_s<\cdots<t_1<1} w_{s_1} \wedge \cdots \wedge w_{s_k},$$

where

$$w_s(t) = \left(\frac{dt}{t}\right)^{\wedge(s-1)} \wedge \frac{dt}{1-t},$$

and therefore qualify already as *naive periods*, as defined in [52].

But in order to understand multiple zeta values as (relative) periods of the cohomology of something, one needs to go one step further and introduce the moduli space $\mathcal{M}_{0,s+3}$ of genus 0 curves with $s+3$ distinct marked points, and its Deligne–Mumford compactification $\bar{\mathcal{M}}_{0,s+3}$.

Indeed, starting from the iterated integral representation, $\zeta(s_1,\ldots,s_k)$ can be shown to be a relative period of a pair

$$(\bar{\mathcal{M}}_{0,s+3} \setminus A, B \setminus (A \cap B))$$

with A and B suitable divisors which have no common irreducible component. These pairs have *mixed Tate* motives, a special (and relatively simple and well-understood) kind of mixed motives. This is a result of Goncharov and Manin [44]. Brown showed that conversely every such relative period of $\bar{\mathcal{M}}_{0,s+3}$ is a rational linear combination of multiple zeta values [25].

Let us now come back to the Feynman periods. Even up to now, not a single example of a Feynman period is known which is not a rational linear combination of multiple zeta values. Moreover, these multiple zeta values do not arise randomly. Rather, certain patterns are visible. For examples of such patterns, see [21], [22], [15], [77].

Motivated by an (informal) conjecture of Kontsevich [50], Belkale and Brosnan investigated the motives associated to Feynman graph hypersurfaces. Kontsevich's conjecture did not state directly that all Feynman periods are multiple zeta values, but that the function

$$q \mapsto |CX_\Gamma(\mathbb{F}_q)|$$

be a polynomial in q for all Γ. Using another conjecture about motives, a non-polynomial counting function for the number of points of CX_Γ over $\mathbb{F}_q$ would imply that CX_Γ has a period which is *not* in the $\mathbb{Q}$-span of multiple zeta values. For example, an elliptic curve is known to have a non-polynomial point counting function.

Belkale and Brosnan came to the surprising result that Kontsevich's conjecture is false [7], and that Feynman graph hypersurfaces have the most general motives one can think of.

4.3 Matroids and Mnëv's theorem. One key idea in Belkale's and Brosnan's proof was to study more general schemes defined by matroids:

Definition 4.1. Let E be a finite set and $I \subseteq 2^E$. The pair $M = (E, I)$ is called a *matroid* if

(1) $\emptyset \in I$,

(2) $A_1 \subseteq A_2, A_2 \in I \implies A_1 \in I$,

(3) $A_1, A_2 \in I, |A_2| > |A_1| \implies$ there is an $x \in A_2 \backslash A_2 \cap A_1$ such that $A_1 \cup \{x\} \in I$.

The number $\operatorname{rk} M = \max_{A \in I} |A|$ is called the *rank of* M.

The subsets $A \in I$ where $|A|$ is maximal are called *bases* of M. The literature usually cites two standard examples for matroids:

(1) $M = (E, I)$ where E is a finite set of vectors in some k^r, I the set of linearly independent subsets of E. Clearly $\operatorname{rk} M \leq r$.

(2) $M = (E, I)$ where E is the set of edges of a graph and I the set of subgraphs (each determined by a subset of edges) without cycles. Clearly $\operatorname{rk} M = |V(\Gamma)| - \operatorname{rk} H_0(\Gamma; \mathbb{Z})$.

We have already seen in Section 2.1 how these examples are related (in fact, the second is a special case of the first): If Γ is a graph, for each $e \in E(\Gamma)$ there is a linear form $e^\vee j_\Gamma$ on $\mathbb{R}^{|V(\Gamma)|}/H_0(\Gamma; \mathbb{R})$, and such linear forms $e_1^\vee j_\Gamma, \ldots, e_n^\vee j_\Gamma$ are pairwise linearly independent if and only if the graph with edges $\{e_1, \ldots, e_n\}$ has no cycles.

Let us return to the general case. A matroid is equivalently characterized by a rank function on 2^E as follows:

Definition 4.2. A map $r : 2^E \to \mathbb{N}$ is called a *rank function* if

(1) $r(A) \leq |A|$,

(2) $A_1 \subseteq A_2 \Longrightarrow r(A_1) \leq r(A_2)$,

(3) $r(A_1 \cup A_2) + r(A_1 \cap A_2) \leq r(A_1) + r(A_2)$.

Proposition 4.1. *Let $M = (E, I)$ be a matroid. Then the map*

$$r : A \mapsto \operatorname{rk}(A, \{B \in I, B \subseteq A\})$$

is a rank function. Conversely, let E be a finite set and r a rank function for it. Then $M = (E, r) = (E, I)$ where $I = \{A \subseteq E, r(A) = |A|\}$ is a matroid. □

We have seen how linearly independent subsets of vectors in a vector space give rise to a matroid. On the other hand one may ask if every matroid is obtained in this way:

Definition 4.3. Let k be a field. A matroid $M = (E, r)$ is called *realizable over* k is there is an $r \in \mathbb{N}$ and a map $f : E \to k^r$ with $\dim \operatorname{span} f(A) = r(A)$ for all $A \in 2^E$. Such a map is called a *representation of* M.

There are matroids which are representable only over certain fields, for example the Fano matroid.

The space $X(M, s)$ of all representations of M in k^s (a subvariety of $\mathbb{A}^{s|E|}$ defined over k) is called the *representation space* of M. It is a fundamental question how general these realization spaces are. An answer is given by Mnëv's Universality Theorem.

Mnëv's Universality Theorem was originally proved by Mnëv in the context of *oriented matroids* and their representations over the ordered field of real numbers. Without giving a precise definition, an oriented matroid keeps track not only of whether or not certain subsets of vectors are linearly dependent but also of the sign of determinants. Roughly, an oriented matroid is specified by a list of partitions of E indicating which vectors in E may be separated by linear hyperplanes in $\mathbb{R}^n$. Again, the representation space of an oriented matroid is the space of vector configurations which leaves this list of partitions invariant. The original, quite difficult, version of the theorem is then

Theorem 4.1 (Mnëv, oriented version). *For every primary semi-algebraic set X in $\mathbb{R}^r$ defined over $\mathbb{Z}$ there is an oriented matroid whose realization space is stably equivalent to X.*

Here a primary semi-algebraic set defined over $\mathbb{Z}$ is a set given by polynomial equations and *sharp* polynomial inequalities $<, >$ with integer coefficients (such as $x_1^2 + x_2^2 > 2$, $x_2 x_1^3 = 1$) and stable equivalence means roughly a sort of homotopy equivalence preserving certain arithmetic properties. The proof in Mnëv's thesis [69], [70] is quite intricate, and there is a simplified proof in [74], [7], which I follow here.

The simpler version that we need is obtained by replacing primary semi-algebraic sets by affine schemes of finite type over $\operatorname{Spec}\mathbb{Z}$, oriented matroids by matroids, and stable equivalence by isomorphism with an open subscheme in a product with $\mathbb{A}^N$. Just like the affine representation space, there is a projective representation space

$$\hat{X}(M,s) = \{f : E \hookrightarrow \mathbb{P}^{s-1} \mid \dim \operatorname{span} f(A) = r(A) - 1 \text{ for all } A \in 2^E\}.$$

Theorem 4.2 (Mnëv, unoriented version). *Let X be an affine scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$. *Then there is a matroid M of rank* 3, *$N \in \mathbb{N}$ and an open $U \subseteq X \times \mathbb{A}^N$ projecting surjectively onto X such that*

$$U \cong \hat{X}(M,3)/\mathrm{PGL}_3.$$

This is the version in Lafforgue's book [65]. I am grateful to A. Usnich for showing me this reference. See also [20] for the independently obtained version of Sturmfels.

Suppose that X is defined by $f_+ - f_- = 0$ where f_+ and f_- are polynomials with positive coefficients. The $f_\pm$ can be successively decomposed into more elementary expressions involving only one addition or one multiplication at a time, at the expense of introducing many more variables. The proof of Theorem 4.2 then uses the fact that once x_1 and x_2 are fixed on a projective line, $x_1 + x_2$ and $x_1 x_2$, etc. can be determined by linear dependence conditions in the projective plane (this is why the rank of M is only 3). The difficulties left are to relate different projective scales and to avoid unwanted dependencies.

In this way any affine scheme over $\operatorname{Spec}\mathbb{Z}$ is related to the representation space of a (huge) rank 3 matroid. Belkale and Brosnan use a slightly different version of Mnëv's theorem and then show (a lot of work that I just skip) how this representation space is connected to the graph hypersurfaces CX_Γ.

Let me now state the main result of [7]: Let GeoMot^+ be the abelian group with generators isomorphism classes $[X]$ of schemes X of finite type over $\mathbb{Z}$ modulo the relation

$$[X] = [X \setminus V] + [V]$$

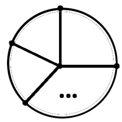
if V is a closed subscheme of X. Endowed with the cartesian product $[X][Y] = [X \times Y]$, GeoMot^+ becomes a ring with unit $[\mathrm{Spec}\,\mathbb{Z}]$. Let $L = [\mathbb{A}^1]$ be the Tate motive, and S the saturated multiplicative subset of $\mathbb{Z}[L]$ generated by $L^n - L$ for $n > 1$. Let $\mathrm{GeoMot} = S^{-1}\mathrm{GeoMot}^+$, and Graphs be the $S^{-1}\mathbb{Z}[L]$-submodule of GeoMot generated by the $[CX_\Gamma]$, where Γ are Feynman graphs.

Theorem 4.3 (Belkale, Brosnan). Graphs = GeoMot. □

It is clear that point-counting $q \mapsto |X(\mathbb{F}_q)|$ factors through GeoMot. Therefore Kontsevich's conjecture is false. Also, it is known [7], Section 15, that the mixed Tate property can be detected in GeoMot. Therefore it follows that not all X_Γ are mixed Tate, and (using another conjecture) that not all periods of all X_Γ are rational linear combinations of multiple zeta values.

On the other hand, not all periods of all X_Γ are Feynman periods in the sense defined in Section 4.2.

4.4 The work of Bloch, Esnault and Kreimer. A finer study of motives of certain Feynman graph hypersurfaces is carried out in the second part of [15]: For the so called *wheels with n spokes*,

$WS_n =$  ,

one has

Theorem 4.4 (Bloch, Esnault, Kreimer).

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X_{WS_n}) \cong \mathbb{Q}(-2),\, H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_{WS_n}) \cong \mathbb{Q}(-2n+3)$$

and $H_{\mathrm{dR}}^{2n-1}(\mathbb{P}^{2n-1} \setminus X_{WS_n})$ *is generated by* $\Omega/\Psi^2_{WS_n}$.

It had been known before [21], [22] that

$$\mathrm{res}_S\, WS_n \in \zeta(2n-3)\mathbb{Q}^\times,$$

and Theorem 4.4 partially confirms that an extension

$$0 \to \mathbb{Q}(2n-3) \to E \to \mathbb{Q}(0) \to 0$$

is responsible for this (see [15], Section 9, [14], Section 9).

4.5 The work of Bloch and Kreimer on renormalization. Let us return to renormalization. Within the parametric Feynman rules, Bloch and Kreimer [16] show how to understand renormalized non-primitive integrals using periods of a *limiting mixed Hodge structure.*

Limiting mixed Hodge structures arise in a situation where there is a family of Hodge structures varying over a base space, in this case a punctured disk D^* (For zero momentum transfer Feynman graphs this one-dimensional base space is sufficient). In contrast to Section 3.4, the parameter $t \in D^*$ does not alter the exponent of the differential form, but is rather some sort of cut-off for the chain of integration.

It follows from our discussion in 3.4 that the projective integral

$$\int_\sigma \frac{\Omega}{\Psi_\Gamma^2}$$

is not convergent unless Γ is primitive (This is the reason why $\operatorname{res}_S \Gamma$ is defined only for primitive integrals): there are poles along the exceptional divisors $\mathcal{E}_{L_\gamma}$ corresponding to divergent subgraphs γ. In other words, $\int_\sigma \frac{\Omega}{\Psi_\Gamma^2}$ is not a period. But by varying the coordinate divisor Δ_t (and the simplex σ_t sitting inside Δ_t) with $t \in D^*$, one has a *family* of mixed Hodge structures, and for all $t \neq 0$ the period $\int_{\sigma_t} \frac{\Omega}{\Psi_\Gamma^2}$ is defined.

Bloch and Kreimer describe how to express the monodromy operation on (relative) homology, in particular on σ_t, in terms of suitable tubes around the strata of the exceptional divisor of $\mathcal{Z}_S$. Winding around such a tube picks up the residue along the stratum (see Section 3.4). Since the monodromy is quasi-unipotent, its logarithm gives a (graph-independent) nilpotent matrix N such that

$$\int_{\sigma_t} \frac{\Omega}{\Psi_\Gamma^2} = \text{first row of } \exp(N \log t/2\pi i)(a_1, \dots, a_r)^t \tag{24}$$

up to a multivalued analytic function vanishing at $t = 0$, with $a_1, \dots, a_r$ periods of a limiting mixed Hodge structure [16].

If there is only one non-zero external momentum, say P, and a certain condition on the momentum flow is satisfied (see [16]) then the relation between the regularization (24) and the renormalized integral (17), where the second graph polynomial must be taken into account, is easy to see. Therefore (24) also tells about the coefficients $p_n(\Gamma)$ of the renormalized integral (17), and one observes in the monodromy representation the same combinatorial objects (nested sets, the Connes–Kreimer coproduct) that have guaranteed locality of counterterms in Section 3.

4.6 Final remarks. Let me finish this second part of the article by just mentioning very briefly some other results that have been obtained in this area.

The Belkale–Brosnan theorem does not provide a specific counterexample graph to Kontsevich's conjecture (it does provide a counterexample matroid). See [35], [76] for recent developments in this direction.

The methods of [15] have been extended in [36] to other graphs than the wheels with spokes. Regularization and renormalization in the parametric representation are also discussed in [17], [18], [66].

The relation between Feynman periods and multiple zeta values as periods of the moduli space of stable genus 0 curves is studied much further in [24], [26]. Finally the reader may be interested in [13], [3], [2], [1], [72] for a further study of graph hypersurfaces.

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Introduction to motives

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With an appendix by Matilde Marcolli

Contents

Introduction

Various cohomology theories play a central role in algebraic geometry, these cohomology theories share common properties and can in some cases be related by specific comparison morphisms. A cohomology theory with coefficients in a ring R is given by a contra-variant functor H from the category of algebraic varieties over a field k to the category of graded R-algebras (or more generally to a R-linear tensor category). The functor H should satisfy certain properties, in particular algebraic cycles on a variety X should give rise to elements in $H(X)$ and the structure of algebraic cycles on X together with their intersection product should be reflected in the structure of $H(X)$. Étale cohomology, de Rham cohomology, Betti cohomology and crystalline cohomology are examples of cohomology theories. Abstracting the formal properties shared by these cohomology theories leads to the notion of a Weil cohomology theory for which the above theories provide examples.

The idea of a universal cohomology theory for algebraic varieties led Grothendieck to the formulation of the theory of motives. Heuristically speaking, given an algebraic variety X over a field k, the motive of X should be an essential object underlying the structure shared by $H(X)$ for various cohomology theories and therefore containing the arithmetic information encoded by algebraic cycles on X. In order to develop a theory of motives, one should then construct a contra-variant functor h from the category of algebraic varieties over k to a category $\mathcal{M}(k)$ through which any cohomology theory will factor. Thus for any Weil cohomology theory H, there should be a realization functor Υ_H defined on $\mathcal{M}(k)$ such that for any algebraic variety X one has $H(X) = \Upsilon_H(h(X))$.

In these notes we will concentrate on motives of smooth projective varieties over an arbitrary base field k, these are called pure motives. The construction of the category of pure motives depends on the choice of an equivalence relation on algebraic cycles on varieties over k. We summarize here the main steps of this construction leaving the details to Chapter 2. Given such an equivalence relation $\sim$ satisfying certain properties, it is possible to enlarge the class of morphisms in the category of smooth projective varieties over k in order to include $\sim$-correspondences thereby linearizing it to an additive category $\mathbf{Corr}_\sim(k)$. By taking the pseudo-abelian envelope of $\mathbf{Corr}_\sim(k)$ one obtains the category of effective motives over k, denoted by $\mathbf{Mot}_\sim^{\mathrm{eff}}(k)$. The product in the category of varieties induces a tensor structure in $\mathbf{Mot}_\sim^{\mathrm{eff}}(k)$ with identity $\mathbf{1}_k$ corresponding to $\mathrm{Spec}(k)$. The projective line $\mathbb{P}^1_k$ decomposes in $\mathbf{Mot}_\sim^{\mathrm{eff}}(k)$ as $\mathbf{1}_k \oplus \mathbb{L}_k$ where $\mathbb{L}_k$ is the Lefschetz motive. The category of pure motives $\mathbf{Mot}_\sim(k)$ is obtained from $\mathbf{Mot}_\sim^{\mathrm{eff}}(k)$ by formally inverting $\mathbb{L}_k$. The functor h from the category of smooth projective varieties over k to $\mathbf{Mot}_\sim(k)$ obtained by composition of the above embeddings is called the functor of motivic cohomology. Some of the properties of the category $\mathbf{Mot}_\sim(k)$ and the extend to which the category depends on the choice of $\sim$ remain largely conjectural. Particular conjectures relating algebraic cycles to cohomology theories, known as the standard conjectures, were introduced by Grothendieck in the sixties partly aiming at giving the basis for the theory of motives (see [37]). The validity of these conjectures would in particular imply that the functor h is itself a cohomology theory.

The contents of this article is as follows. After recalling the necessary background in Chapter 1 we review the main steps of the construction of the category of pure motives in Chapter 2. Chapter 3 is devoted to Artin motives. A quick view of more advanced topics in Chapter 4 precedes a review of the theory of endomotives in Chapter 5. The second part of the article consist of an extended appendix (by Matilde Marcolli) surveying the role of motives in noncommutative geometry.

1 Preliminaries

Throughout this section we fix a base field k.

1.1 Cycles and correspondences. Let X be an smooth projective variety over k. A *prime algebraic cycle* on X is by definition a closed irreducible subvariety of X. Denote by $C(X)$ the free abelian group generated by prime algebraic cycles on X and by $C^r(X)$ the subgroup of $C(X)$ generated by prime algebraic cycles of codimension r in X. An element $Z \in C(X)$ is called an *algebraic cycle*, if $Z \in C^r(X)$ we say that Z is an algebraic cycle of codimension r on X. Any cycle $Z \in C^r(X)$ can therefore be written as a finite formal linear combination $Z = \sum n_i Z_i$ where $n_i \in \mathbb{Z}$ and each Z_i is a closed irreducible subvariety of X of codimension r.

Let Z_1 and Z_2 be two prime cycles on X. We say that Z_1 and Z_2 intersect properly if $\operatorname{codim}(Z_1 \cap Z_2) = \operatorname{codim}(Z_1) + \operatorname{codim}(Z_2)$. In this case we can define an algebraic cycle $Z_1 \bullet Z_2$ of codimension $\operatorname{codim}(Z_1) + \operatorname{codim}(Z_2)$ on X as a linear combination of the irreducible components of $Z_1 \cap Z_2$ with coefficients given by the intersection multiplicities (cf. [23]). More generally two algebraic cycles $Z_1, Z_2 \in C(X)$ intersect properly if every prime cycle in Z_1 intersects properly with every prime cycle in Z_2 in which case we obtain a well defined cycle $Z_1 \bullet Z_2$ extending by linearity the above definition.

The intersection product $\bullet$ gives a partially defined multiplication from $C^r(X) \times C^{r'}(X)$ to $C^{r+r'}(X)$ which is compatible with the abelian group structure on $C(X)$. In order to obtain a graded ring starting from cycles and reflecting the geometric properties of their intersections it is necessary to impose an appropriate equivalence relation in such a way that $\bullet$ induces a well defined multiplication. There are various possible choices for such an equivalence relation leading to corresponding rings of cycles. Before analyzing these in a more systematic way it is useful to study the functoriality properties of algebraic cycles.

Let $\varphi\colon X \to Y$ be a morphism between two smooth projective varieties over k. Let Z be a prime cycle on X. Since φ is proper $W = \varphi(Z)$ is a closed irreducible subvariety of Y. If $\dim Z = \dim W$ then the function field $k(Z)$ is a finite extension of $k(W)$. Let $d = [k(Z) : k(W)]$ be the degree of the extension $k(W) \hookrightarrow k(Z)$ if $\dim Z = \dim W$ and set $d = 0$ otherwise. Then the map $\varphi_*\colon Z \mapsto d\,W$ extends by linearity to a group homomorphism

$$\varphi_*\colon C^r(X) \to C^{r+(n-m)}(Y),$$

where $m = \dim X$ and $n = \dim Y$. We call φ_* the push-forward of φ.

For φ as above let $\Gamma(\varphi) \subset X \times Y$ denote the graph subvariety of φ. If W is a cycle in Y such that $(X \times W) \bullet \Gamma(\varphi)$ is defined we identify this product with a cycle $Z = \varphi^*(W)$ on X via the isomorphism $X \simeq \Gamma(\varphi)$. If W is a prime cycle on Y then $\varphi^*(W)$ is a linear combination of the irreducible components of $\varphi^{-1}(W)$. Moreover, if φ is flat of constant relative dimension then $\varphi^*(W) = \varphi^{-1}(W)$. The operator φ^* is linear and multiplicative whenever the appropriate cycles are defined. We call φ^* the pull-back of φ.

The two maps φ^* and φ_* are related by the projection formula

$$\varphi_*(\varphi^*(W) \bullet Z) = W \bullet \varphi_*(Z),$$

which holds for any cycles Z on X and W on Y for which $\varphi^*(W) \bullet Z$ and $W \bullet \varphi_*(Z)$ are defined.

Let $\sim$ be an equivalence relation defined on algebraic cycles on smooth projective varieties over k. The equivalence relation $\sim$ is called *adequate* if it satisfies the following properties (cf. [48], [36], [31]):

- The equivalence relation $\sim$ is compatible with addition of cycles.
- If Z_1 and Z_2 are two algebraic cycles on X then there exists a cycle $Z_2' \in C(X)$ such that $Z_2 \sim Z_2'$ and Z_1 intersects properly with Z_2'.
- Let X and Y be two smooth projective varieties over k. Denote by pr_2 the projection morphism from $X \times Y$ to Y. Let Z be a cycle on X and W be a cycle on $X \times Y$ such that $W \bullet (Z \times Y)$ is defined. Then $Z \sim 0$ in $C(X)$ implies $(\mathrm{pr}_2)_*(W \bullet (Z \times Y)) \sim 0$ in $C(Y)$.

In short $\sim$ is an adequate equivalence relation if pull-back, push-forward and intersection of cycles are well defined modulo $\sim$. If $\sim$ is an adequate equivalence relation on cycles then for any smooth projective variety X over k the residue classes of $\sim$ form a ring under intersection product:

$$A_\sim(X) := C(X)/\sim\, = \bigoplus_r A^r_\sim(X),$$

where $A^r_\sim(X) := C^r(X)/\sim$. Given a morphism $\varphi\colon X \to Y$ of smooth projective varieties the pull-back and push-forward operations on cycles induce a multiplicative operator

$$\varphi^*\colon A_\sim(Y) \to A_\sim(X)$$

and an additive operator

$$\varphi_*\colon A^r_\sim(X) \to A^{r+(n-m)}_\sim(Y)$$

where $m = \dim X$ and $n = \dim Y$.

Example 1.1. Let X be a smooth projective variety over k. Two cycles Z_1 and Z_2 on X are said to be rationally equivalent if there exists an algebraic cycle W on $X \times \mathbb{P}^1$ such that Z_1 is the fiber of W over 0 and Z_2 is the fiber of W over 1. We denote the resulting equivalence relation on cycles by $\sim_{\mathrm{rat}}$. The fact that $\sim_{\mathrm{rat}}$ is an adequate equivalence relation is a consequence of Chow's moving lemma (see [36], [23]). For a variety X the ring $A_{\mathrm{rat}}(X)$ is called the Chow ring of X.

Example 1.2. Let X be a smooth projective variety over k. Two cycles Z_1 and Z_2 on X are said to be algebraically equivalent if there exists an irreducible curve T over k and a cycle W on $X \times T$ such that Z_1 is the fiber of W over t_1 and Z_2 is the fiber of W over t_2 for two points $t_1, t_2 \in T$. We denote the resulting equivalence relation on cycles by $\sim_{\mathrm{alg}}$. As above $\sim_{\mathrm{alg}}$ is an adequate equivalence relation.

Example 1.3. Let H be a Weil cohomology theory on smooth projective varieties over k with coefficients in a field F of characteristic 0 (see Section 1.2). Let X be a smooth projective variety over k with corresponding cycle class map cl_X. Two cycles $Z_1, Z_2 \in C^r(X)$ are said to be*homologically equivalent* with respect to H if $\mathrm{cl}_X(Z_1) = \mathrm{cl}_X(Z_2)$. We denote the resulting equivalence relation on cycles by $\sim_{\mathrm{hom}}$. Homological equivalence is an adequate equivalence relation for any Weil cohomology theory.

Example 1.4. Let X be a smooth projective variety of dimension n over k. Two cycles $Z_1, Z_2 \in C^r(X)$ are said to be numerically equivalent if for any $W \in C^{n-r}(X)$ for which $Z_1 \bullet W$ and $Z_2 \bullet W$ are defined we have $Z_1 \bullet W = Z_2 \bullet W$. We denote the resulting equivalence relation on cycles by $\sim_{\mathrm{num}}$. Numerical equivalence is an adequate equivalence relation.

Remark 1.5 (cf. [36], [31]). Let X be a smooth projective variety over k. Given any adequate equivalence relation $\sim$ on algebraic cycles on smooth projective varieties over k there exist canonical morphisms

$$A_{\mathrm{rat}}(X) \to A_\sim(X)$$

and

$$A_\sim(X) \to A_{\mathrm{num}}(X).$$

Rational equivalence is therefore the finest adequate equivalence relation for algebraic cycles on smooth projective varieties over k. Likewise numerical equivalence is the coarsest (non-zero) adequate equivalence relation for algebraic cycles on smooth projective varieties over k.

The following simple result will be used later (see [36], [42]):

Lemma 1.6. *Let $\sim$ be an adequate equivalence relation on algebraic cycles on smooth projective varieties over k. Choose a rational point in $\mathbb{P}^1_k$ and denote by e its class modulo $\sim$. Then*

$$A_\sim(\mathbb{P}^1_k) = \mathbb{Z} \oplus \mathbb{Z}e$$

1.2 Weil cohomology theories. Denote by $\mathcal{V}(k)$ the category of smooth projective varieties over k. Let F be a field of characteristic 0 and denote by GrAlg_F the category of graded F-algebras. Consider a contravariant functor

$$H\colon \mathcal{V}(k)^{\mathrm{op}} \to \mathrm{GrAlg}_F, \quad X \mapsto H(X) = \bigoplus_{r \geq 0} H^r(X).$$

For any morphism $\varphi\colon X \to Y$ in $\mathcal{V}(k)$ denote $H(\varphi)\colon H(Y) \to H(X)$ by φ^*. Assume moreover that there exist a covariant operator from morphisms $\varphi\colon X \to Y$ to linear maps $\varphi_*\colon H(X) \to H(Y)$. The fact that we use the same notations than for the induced pull-back and push-forward maps at the level of cycles should not cause any confusion. A functor H as above is a *Weil cohomology theory with coefficients in F* if it satisfies the following properties (cf. [36]):

- Given a variety X in $\mathcal{V}(k)$ the space $H^r(X)$ is finite dimensional for any $r \geq 0$. If $r > 2\dim X$ then $H^r(X) = 0$.
- Let $P = \mathrm{Spec}(k)$. Then there exist an isomorphism $a\colon H(P) \to F$.
- For any morphism $\varphi\colon X \to Y$ in $\mathcal{V}(k)$ and any $x \in H(X)$, $y \in H(Y)$ the projection formula
$$\varphi_*(\varphi^*(y)x) = y\varphi_*(x)$$
holds.
- For any X and Y in $\mathcal{V}(k)$,
$$H(X \amalg Y) \simeq H(X) \oplus H(Y).$$
- For any X and Y in $\mathcal{V}(k)$ the *Künneth formula*
$$H(X \times Y) \simeq H(X) \otimes H(Y)$$
holds.
- For any X in $\mathcal{V}(k)$ let $\varphi_X : P = \mathrm{Spec}(k) \to X$ be the structure morphism and define the degree map $\langle\ \rangle\colon H(X) \to F$ as the composition $a \circ \varphi_X^*$. Then the pairing
$$H(X) \otimes H(X) \to F, \quad x_1 \otimes x_2 \mapsto \langle x_1 x_2\rangle,$$
is nondegenerate.
- For every smooth projective variety over k there exist a group homomorphism $\mathrm{cl}_X : C(X) \to H(X)$, the *cycle class map*, satisfying:
  (1) The map
  $$\mathrm{cl}_{\mathrm{Spec}(k)} : C(\mathrm{Spec}(k)) \simeq \mathbb{Z} \to H(\mathrm{Spec}(k)) \simeq F$$
  is the canonical homomorphism.
  (2) For any morphism $\varphi\colon X \to Y$ in $\mathcal{V}(k)$ its pull-back and push-forward commute with the cycle class map:
  $$\varphi^* \mathrm{cl}_Y = \mathrm{cl}_X \varphi^*, \quad \varphi_* \mathrm{cl}_X = \mathrm{cl}_Y \varphi_*.$$
  (3) Let X and Y be varieties in $\mathcal{V}(k)$, let $Z \in C(X)$ and $W \in C(Y)$. Then
  $$\mathrm{cl}_{X\amalg Y}(Z \amalg W) = \mathrm{cl}_X(Z) \oplus \mathrm{cl}_Y(W),$$
  $$\mathrm{cl}_{X\times Y}(Z \times W) = \mathrm{cl}_X(Z) \otimes \mathrm{cl}_Y(W).$$

Example 1.7. Let k be a field of characteristic zero together with an embedding $k \hookrightarrow \mathbb{C}$. The Betti cohomology of a variety X in $\mathcal{V}(k)$ is defined as the singular cohomology of $X(\mathbb{C})$ with coefficients in $\mathbb{Q}$. Betti cohomology is a Weil cohomology theory with coefficients in $\mathbb{Q}$.

Example 1.8. Let k be a field of characteristic zero. The de Rham cohomology of a variety X over k can be defined in terms of the hypercohomology of its algebraic de Rham complex (see [28]). De Rham cohomology is a Weil cohomology theory with coefficients in k.

Example 1.9. Let k be a field of characteristic $p > 0$ and let $l \neq p$ be a prime number. The étale cohomology of a variety X in $\mathcal{V}(k)$ is defined as the l-adic cohomology of $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$. Étale cohomology is a Weil cohomology theory with coefficients in $\mathbb{Q}_l$.

Example 1.10. Let k be a field of characteristic $p > 0$. Crystalline cohomology was introduced by Grothendieck and developed by Berthelot as a substitute for l-adic étale cohomology in the $l = p$ case (see [29]). Let $W(k)$ be the ring of Witt vectors with coefficients in k and let $F_{\mathrm{Witt}(k)}$ be its field of fractions. Crystalline cohomology is a Weil cohomology theory with coefficients in $F_{\mathrm{Witt}(k)}$.

Remark 1.11. It is possible to define cohomology theories in a more general setting where the functor H takes values on a linear tensor category $\mathcal{C}$. Properties analogous to the aforementioned ones should then hold. In particular H should be a symmetric monoidal functor from $\mathcal{V}(k)$ to $\mathcal{C}$, where we view $\mathcal{V}(k)$ as a symmetric monoidal category with product $X \times Y = X \times_{\mathrm{Spec}\, k} Y$ (this is just the Künneth formula).

1.3 Correspondences. Let $\sim$ be a fixed adequate equivalence relation on algebraic cycles on smooth projective varieties over k.

Definition 1.12. Let X and Y be two varieties in $\mathcal{V}(k)$. An element $f \in A_\sim(X \times Y)$ is called a *correspondence* between X and Y.

Note that this definition depends on the choice of the adequate equivalence relation $\sim$.

Given varieties X_1, X_2 and X_3 denote by

$$\mathrm{pr}_{i,j} \colon X_1 \times X_2 \times X_3 \to X_i \times X_j, \quad 1 \leq i < j \leq 3,$$

the projection morphisms. Given two correspondences $f \in A_\sim(X_1 \times X_2)$ and $g \in A_\sim(X_2 \times X_3)$ we define their composition as the correspondence

$$g \circ f = \mathrm{pr}_{1,3\,*}(\mathrm{pr}_{1,2}^*(f)\,\mathrm{pr}_{2,3}^*(g)) \in A_\sim(X_1 \times X_3).$$

It can be shown that composition of correspondences is associative for any adequate equivalence relation (cf. [42]).

For a variety X in $\mathcal{V}(k)$ we denote by Δ_X the class of the diagonal cycle $X \hookrightarrow X \times X$ in $A_\sim(X \times X)$. If X and Y are two varieties in $\mathcal{V}(k)$ and $f \in A_\sim(X \times Y)$, $g \in A_\sim(Y \times X)$ are correspondences, then

$$f \circ \Delta_X = f, \quad \Delta_X \circ g = g.$$

Thus Δ_X behaves as the identity with respect to composition of correspondences.

For any morphism $\varphi\colon X \to Y$ denote by $\widetilde{\Gamma}_\varphi$ the correspondence in $A_\sim(X \times Y)$ given by the class of the cycle $\Gamma_\varphi \hookrightarrow X \times Y$ given by the graph of φ.

2 From varieties to pure motives

As in the previous section let k be a fixed base field and denote by $\mathcal{V}(k)$ the category of smooth projective varieties over k. Throughout this section we fix an adequate equivalence relation $\sim$ for algebraic cycles on varieties in $\mathcal{V}(k)$. In this section we will use the formalism developed in Section 1.3 to "linearize" the category $\mathcal{V}(k)$.

2.1 Linearization. Correspondences between varieties possess all the formal properties of morphisms, we can therefore construct a new category whose objects correspond to smooth projective varieties over k but whose morphisms are given by correspondences. Since cycles modulo an adequate equivalence relation form an abelian group, the category thus obtained will have the advantage of being an additive category.

Given two varieties X and Y in $\mathcal{V}(k)$ we set

$$\mathbf{Corr}_\sim(X, Y) = \bigoplus_i A_\sim^{\dim X_i}(X \times Y)$$

where X_i are the irreducible components of X.

Definition 2.1. Let F be a field of characteristic 0. We define $\mathbf{Corr}_\sim(k, F)$, the category of correspondences over k with coefficients in F, as the category whose objects are smooth projective varieties over k,

$$\mathrm{Obj}(\mathbf{Corr}_\sim(k, F)) = \mathrm{Obj}(\mathcal{V}(k)),$$

and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{Corr}_\sim(k,F)}(X, Y) = \mathbf{Corr}_\sim(X, Y) \otimes F.$$

Composition of morphisms is given by composition of correspondences. The identity morphism in $\mathrm{Hom}_{\mathbf{Corr}(k)}(X, X)$ is given by the correspondence Δ_X.

It follows from the definitions that given any adequate equivalence relation $\sim$ for algebraic cycles on varieties in $\mathcal{V}(k)$ the category $\mathbf{Corr}_\sim(k, F)$ is an F-linear category. We denote $\mathbf{Corr}_\sim(k, \mathbb{Q})$ by $\mathbf{Corr}_\sim(k)$ and refer to it simply as the category of correspondences over k.

The category $\mathcal{V}(k)$ can be faithfully embedded into the category $\mathbf{Corr}_\sim(k, F)$ via the contravariant functor

$$h\colon \mathcal{V}(k) \to \mathbf{Corr}_\sim(k, F),$$

which acts as the identity on objects and sends a morphism $\varphi\colon Y \to X$ in $\mathcal{V}(k)$ to the correspondence $\widetilde{\Gamma}_\varphi$ in $\mathrm{Hom}_{\mathbf{Corr}_\sim(k,F)}(X, Y)$ (see 1.3). We denote therefore by $h(X)$ a smooth projective variety X when considered as an object in $\mathbf{Corr}_\sim(k, F)$.

The product of varieties in $\mathcal{V}(k)$ induces a tensor structure in the category $\mathbf{Corr}_\sim(k, F)$ via

$$h(X) \otimes h(Y) = h(X \times Y)$$

turning $\mathbf{Corr}_\sim(k, F)$ into a F-linear tensor category with identity object given by

$$\mathbf{1}_k := h(\operatorname{Spec} k).$$

2.2 Pseudo-abelianization. The category $\mathbf{Corr}_\sim(k, F)$ obtained in Section 2.1, although being F-linear, is still far from abelian. In particular not every idempotent morphism in $\mathbf{Corr}_\sim(k, F)$ corresponds to a direct sum decomposition of the underlying object. In this section, we will formally add the kernels of idempotent morphisms in $\mathbf{Corr}_\sim(k, F)$ in order to obtain a pseudo-abelian category. As this formal procedure can be carried out for any additive category, we start this section by describing it in this generality.

Definition 2.2. An additive category $\mathcal{A}$ is called pseudo-abelian if for any object A in $\mathcal{A}$ and any idempotent endomorphism $p = p^2 \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ there exist a kernel $\ker p$ and the canonical morphism:

$$\ker p \oplus \ker(\mathrm{id}_A - p) \to A$$

is an isomorphism.

Given any additive category $\mathcal{D}$, it is possible to construct a pseudo-abelian category $\widetilde{\mathcal{D}}$ into which $\mathcal{D}$ embeds fully faithfully via a functor

$$\Psi_{\mathcal{D}}\colon \mathcal{D} \to \widetilde{\mathcal{D}},$$

which is universal in the sense that given any additive functor $F\colon \mathcal{D} \to \mathcal{A}$, where $\mathcal{A}$ is a pseudo-abelian category, there exists an additive functor $\widetilde{F}\colon \widetilde{\mathcal{D}} \to \mathcal{A}$ such that the functors G and $\widetilde{F}\Psi_{\mathcal{D}}$ are equivalent.

The category $\widetilde{\mathcal{D}}$ is obtained by formally adding kernels of idempotent endomorphisms in $\mathcal{D}$. Objects in the category $\widetilde{\mathcal{D}}$ are given by pairs (D, p) where D is an object in $\mathcal{D}$ and $p = p^2 \in \mathrm{Hom}_{\mathcal{D}}(D, D)$ is an idempotent endomorphism:

$$\mathrm{Obj}(\widetilde{\mathcal{D}}) = \{(D, p) \mid D \in \mathrm{Obj}(\mathcal{D}),\ p = p^2 \in \mathrm{Hom}_{\mathcal{D}}(D, D)\}.$$

If (D, p) and (D', p') are objects in $\widetilde{\mathcal{D}}$ we define $\mathrm{Hom}_{\widetilde{\mathcal{D}}}((D, p), (D', p'))$ to be the quotient group

$$\frac{\{f \in \mathrm{Hom}_{\mathcal{D}}(D, D') \text{ such that } fp = p'f\}}{\{f \in \mathrm{Hom}_{\mathcal{D}}(D, D') \text{ such that } fp = p'f = 0\}}.$$

Composition of morphisms is induced from composition of morphisms in $\mathcal{D}$. The category thus obtained is a pseudo-abelian category and the functor given on objects by

$$\Psi_{\mathcal{D}} : D \mapsto (D, \mathrm{id}_D)$$

and sending a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ to its class in $\mathrm{Hom}_{\mathcal{D}}(D, D')/(0)$ is fully faithful and satisfies the above universal property.

We call $\widetilde{\mathcal{D}}$ the pseudo-abelian envelope of $\mathcal{D}$ (also sometimes referred to as the idempotent completion of $\mathcal{D}$, or the Karoubi envelope $\mathcal{D}$). If the category $\mathcal{D}$ is an F-linear category for a field F then $\widetilde{\mathcal{D}}$ is an F-linear pseudo-abelian category. If the category $\mathcal{D}$ has an internal tensor product $\otimes$ then the product

$$(D, p) \otimes (D', p') = (D \otimes D', p \otimes p')$$

is an internal tensor product on $\widetilde{\mathcal{D}}$.

Definition 2.3. Let F be a field of characteristic 0. The category of *effective motives* over k with coefficients in F, denoted by $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, F)$, is the pseudo-abelian envelope of the category $\mathbf{Corr}_{\sim}(k, F)$.

As above we denote $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, \mathbb{Q})$ by $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k)$ and refer to its objects simply as *effective motives* over k. The category $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, F)$ is by construction a pseudo-abelian F-linear tensor category.

We can extend the functor h from $\mathcal{V}(k)$ to $\mathbf{Corr}_{\sim}(k, F)$ to a functor from $\mathcal{V}(k)$ to $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, F)$ by composing it with the canonical embedding $\Psi_{\mathbf{Corr}_{\sim}(k,F)}$, we denote the functor thus obtained also by h.

Spelling out the definition of the pseudo-abelian envelope in this particular case, we see that effective motives over k can be represented as pairs

$$(h(X), p)$$

where X is a smooth projective variety over k and $p \in \mathbf{Corr}(X, X) \otimes F$ is an idempotent correspondence. Since for any such idempotent we have

$$\ker p \oplus \ker(\Delta_X - p) = h(X)$$

in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, F)$, we see that effective motives over k are essentially given by direct factors of smooth projective varieties over k.

Consider the case of $\mathbb{P}^1_k$. Let e be as in Lemma 1.6, then the correspondence $(1 \times e) \in \mathbf{Corr}(\mathbb{P}^1_k, \mathbb{P}^1_k)$ is idempotent. We define the *Lefschetz motive* over k to be the effective motive given by

$$\mathbb{L}_k = (h(\mathbb{P}^1_k), (1 \times e));$$

in particular we get a decomposition of $\mathbb{P}^1_k$ in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, F)$ of the form

$$h(\mathbb{P}^1_k) = \mathbf{1}_k \oplus \mathbb{L},$$

where as above we take $\mathbf{1}_k = h(\mathrm{Spec}(k))$. More generally, we obtain a decomposition of r-dimensional projective space over k as

$$h(\mathbb{P}^r) = \mathbf{1} \oplus \mathbb{L}_k \oplus \cdots \oplus \mathbb{L}_k^r,$$

where

$$\mathbb{L}_k^i = \mathbb{L}_k \otimes \cdots \otimes \mathbb{L}_k,\ i \text{ times}.$$

It can also be shown that an irreducible curve X in $\mathcal{V}(k)$ admits a decomposition in $\mathbf{Mot}_\sim^{\mathrm{eff}}(k, F)$ of the form

$$h(X) = \mathbf{1} \oplus h^1(X) \oplus \mathbb{L}_k.$$

2.3 Inversion. Tensoring with the Lefschetz motive induces a functor

$$M \to M \otimes \mathbb{L}_k, \quad f \mapsto f \otimes \mathrm{id}_{\mathbb{L}_k},$$

from the category $\mathbf{Mot}_\sim^{\mathrm{eff}}(k, F)$ to itself. This functor is fully faithful. In particular, given two effective motives M and M' and integers n, m, N with $N \geq n, m$ the F-vector space

$$\mathrm{Hom}_{\mathbf{Mot}_\sim^{\mathrm{eff}}(k,F))}(M \otimes \mathbb{L}_k^{N-m}, M' \otimes \mathbb{L}_k^{N-n})$$

is independent of the choice of N. This can be used to obtain the category of pure motives from the category of effective motives by formally inverting the element $\mathbb{L}_k$. More precisely, define the category of *pure motives* $\mathbf{Mot}_\sim(k, F)$ as the category whose objects are given by pairs (M, n), where M is an effective motive and m is an integer,

$$\mathrm{Obj}(\mathbf{Mot}_\sim(k, F)) = \{(M, m) \mid M \in \mathrm{Obj}(\mathbf{Mot}_\sim^{\mathrm{eff}}(k, F)),\ m \in \mathbb{Z}\}$$

and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{Mot}_\sim(k,F)}((M, m), (M', n)) = \mathrm{Hom}_{\mathbf{Mot}_\sim^{\mathrm{eff}}(k,F))}(M \otimes \mathbb{L}_k^{N-m}, M' \otimes \mathbb{L}_k^{N-n}),$$

where $N \geq n, m$. As above we let $\mathbf{Mot}_\sim(k, \mathbb{Q}) = \mathbf{Mot}_\sim(k)$.

The category $\mathbf{Mot}_\sim(k, F)$ has a tensor product given by

$$(M, m) \otimes (M', n) = (M \otimes M', m + n).$$

We can embed the category of effective motives $\mathbf{Mot}_\sim^{\mathrm{eff}}(k, F))$ as a subcategory of $\mathbf{Mot}_\sim(k, F)$ via the functor

$$M \mapsto (M, 0).$$

As before we denote by h the functor from $\mathcal{V}(k)$ to $\mathbf{Mot}_\sim(k, F)$ induced by the above embedding.

Denote by $\mathbb{T}_k$ the object $(\mathbf{1}_k, -1)$ in $\mathbf{Mot}_\sim(k, F)$ and write $\mathbb{T}_k^n$ for $(\mathbf{1}_k, -n), n \in \mathbb{Z}$. Then $\mathbb{T}_k^0 = \mathbf{1}_k$ and there is a canonical isomorphism

$$\mathbb{T}_k^{-1} = \mathbb{L}_k.$$

The element $\mathbb{T}_k$ is called the *Tate motive*. $\mathbb{T}_k$ plays a role analogous to the Tate module in l-adic cohomology. We define

$$M(n) = M \otimes \mathbb{T}_k^n \quad \text{(“Tate twisting”)}.$$

Any pure motive can be written as $M(n)$ for an effective motive M and an integer n. It can be shown (see [22]) that for any smooth projective variety X there are canonical isomorphisms

$$A^r_\sim(X) \otimes F \simeq \mathrm{Hom}_{\mathbf{Mot}_\sim(k,F)}(\mathbf{1}_k, h(X)(r)).$$

As mentioned in the Introduction some of the properties of the category $\mathbf{Mot}_\sim(k)$ and the extent to which it depends on the choice of $\sim$ remain largely conjectural. We end this section with an important result due to Jannsen in the case $\sim$ is numerical equivalence on cycles (Example 1.4).

Theorem 2.4 (Jannsen [30]). *The category* $\mathbf{Mot}_{\mathrm{num}}(k, F)$ *is a semi-simple* F*-linear, rigid tensorial category.*

3 Artin motives

Artin motives are motives of zero-dimensional varieties, already at this level various facets of the theory make their appearance and some of the richness of the underlying structures manifest itself. In a sense which we will make more precise, the theory of Artin motives can be considered as a linearization of Galois theory.

Let $\mathcal{V}^0(k)$ be the subcategory of $\mathcal{V}(k)$ consisting of varieties of dimension 0 over k. Objects in $\mathcal{V}^0(k)$ are given by spectra of finite k-algebras:

$$\mathrm{Obj}(\mathcal{V}^0(k)) = \{X \in \mathrm{Obj}(\mathcal{V}(k)) \mid X = \mathrm{Spec}(A),\ \dim_k A < \infty\}.$$

Fix a separable closure k^{sep} of k. Then for any $X = \mathrm{Spec}(A) \in \mathrm{Obj}(\mathcal{V}^0(k))$ the absolute Galois group $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts continuously on the set of algebraic points of X:

$$X(k^{\mathrm{sep}}) = \mathrm{Hom}_{k-\mathrm{alg}}(A, k^{\mathrm{sep}}).$$

The action of G_k commutes with morphisms in $\mathcal{V}^0(k)$ since these are given by rational maps. Taking algebraic points induces then a functor

$$X \mapsto X(k^{\mathrm{sep}})$$

between $\mathcal{V}^0(k)$ and the category $\mathcal{F}^0(G_k)$ consisting of finite sets endowed with a continuous G_k-action. The fact that this functor is an equivalence of categories is essentially a restatement of the main theorem of Galois theory. The inverse functor maps a finite G_k-set $I \in \mathrm{Obj}(\mathcal{F}^0(G_k))$ to the spectrum of the ring of G_k-invariant functions from I to k^{sep}. This equivalence of categories is usually referred as the Grothendieck–Galois correspondence.

The category of *Artin motives* $\mathbf{Mot}^0(k,F)$, is by definition the subcategory of $\mathbf{Mot}_\sim(k,F)$ spanned by objects of the form $h(X)$ for $X \in \mathrm{Obj}(\mathcal{V}^0(k))$. It is important to note that since any adequate equivalence relation on algebraic cycles on varieties in $\mathcal{V}(k)$ becomes trivial when restricted to $\mathcal{V}^0(k)$, this definition does not depend on the choice of $\sim$. Also, since the Lefschetz motive corresponds to the decomposition of the one dimensional variety $\mathbb{P}^1_k$ there is no need to take into account the twisting by $\mathbb{T}_k$. The category $\mathbf{Mot}^0(k,F)$ is therefore the pseudo-abelian envelope of the category of correspondences of zero-dimensional varieties $\mathbf{Corr}^0(k,F)$, whose objects are given by varieties in $\mathcal{V}^0(k)$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{Corr}^0(k,F)}(X,Y) = C^0(X \times Y) \otimes F.$$

A correspondence between two varieties X and Y in $\mathcal{V}^0(k)$ (with coefficients in F) is thus given by a formal linear combination of connected components of $X \times Y$ with coefficients in F. By taking characteristic functions we may identify such a correspondence with a G_k-invariant function from $X(k^{\mathrm{sep}}) \times Y(k^{\mathrm{sep}})$ to F. Composition of correspondences becomes matrix multiplication and passing to the pseudo–abelian envelope we get an equivalence of categories,

$$\mathbf{Mot}^0(k,F) \simeq \mathbf{Rep}(G_k,F),$$

where $\mathbf{Rep}(G_k,F)$ is the category of finite dimensional F-representations of the group G_k. The functor of motivic cohomology restricted to dimension zero is then given by

$$h\colon X \mapsto F^{X(k^{\mathrm{sep}})},$$

where $X \in \mathrm{Obj}(\mathcal{V}^0(k))$ and the F-vector space $F^{X(k^{\mathrm{sep}})}$ is endowed with the natural G_k-action. The category $\mathbf{Mot}^0(k,F)$ has a rich structure coming from the fact that it can be identified with the category of representations of a group, the corresponding properties encoded thereby correspond to the fact that $\mathbf{Mot}^0(k,F)$ is a Tannakian category.

When reference to an ambient category is relevant it is customary to view $\mathbf{Mot}^0(k,F)$ as a subcategory of $\mathbf{Mot}_{\mathrm{num}}(k,F)$.

4 Vistas

As mentioned in the introduction, the formal properties in the construction of $\mathbf{Mot}_\sim(k,F)$ imply the existence of realization functors

$$\Upsilon_H\colon\ \mathbf{Mot}_\sim(k,F) \to \mathrm{GrAlg}_F$$

for various Weil cohomology theories $H\colon \mathcal{V}(k)^{\mathrm{op}} \to \mathrm{GrAlg}_F$ defined on $\mathcal{V}(k)$. These functors enrich the structure of $\mathbf{Mot}_\sim(k,F)$ and lead to further important constructions.

Realization functors play an important role in the definition of L-functions associated to motives the special values of which are of prime importance in number theory (cf. [21]).

Realization functors also play a role in the definition of motivic Galois groups. In order to be able to modify the category $\mathbf{Mot}_{\sim}(k, F)$ to obtain a category equivalent to the category of representations of a group,it is necessary to have a fiber functor with values on F-vector spaces playing the role of forgetful functor. As mentioned in the previous section in the case of Artin motives the absolute Galois group of the base field is recovered from this formalism. However more general cases involve in a deep way the validity of the standard conjectures (see [50] for a review). Once a Tannakian category of motives has been constructed the corresponding motivic Galois group provides a rich higher dimensional analogue of Galois theory.

From here on, the theory develops rapidly and branches in numerous directions, leading to a very rich landscape of results connecting and interrelating various areas of mathematics. The largely conjectural theory of mixed motives, that is motives of varieties which are not necessary smooth or projective, seems to underlay phenomena relevant to different fields. Areas like Hodge theory, K-theory and automorphic forms are enriched by the presence of structures of motivic origin leading in many cases to deep conjectures. For an account of various aspects of the theory the reader may consult the two volumes [32].

5 Endomotives

Various interactions between noncommutative geometry and number theory have taken place recently. The tools and methods developed by noncommutative geometry are well suited for the study of structures relevant in number theory, and in many cases shed new light into old outstanding problems (cf. [44], [15]). Some aspects of the theory so developed make contact with the theory of motives in a natural way. In this section we describe the construction of a class of noncommutative spaces closely related to Artin motives and its relevance in the study of class field theory. The main reference for the material treated in this section is [11], which we follow closely (see also [15], [18]).

5.1 Adèles and idèles, basics in class field theory. Global class field theory describes the abelian extensions of a global field k in terms of analytic-arithmetic data coming from the field itself. We start this section by briefly recalling some basic notions from global class field theory (cf. [1]). A global field k is by definition a field of one of the following two kinds:

- a number field, i.e., a finite extension of $\mathbb{Q}$, the field of rational numbers;
- a function field, i.e., a finite extension of $\mathbb{F}_q(t)$, the field of rational functions in one variable over a finite field.

An important part of the structure of a global field k is encoded by ideals in its ring of integers $\mathcal{O}_k$ which is given by the integral closure in k of $\mathbb{Z}$ in the number field case or that of $\mathbb{F}_q[t]$ in the function field case.

A *valuation* on a global field k is by definition a nonnegative multiplicative function $|\cdot|$ from k to $\mathbb{R}$, with $|0| = 0$ and non-vanishing on $k^* = k \setminus \{0\}$, satisfying the triangle inequality

$$|x + y| \le |x| + |y| \quad \text{for all } x, y \in k.$$

The valuation $|\cdot|$ is called *non-archimedean* if it satisfies

$$|x + y| \le \max\{|x|, |y|\} \quad \text{for all } x, y \in k,$$

otherwise we say that the valuation $|\cdot|$ is archimedean. Two valuations on a global field k are said to be equivalent if the corresponding metrics induce the same topology on k. A *place* ν on a global field k is by definition an equivalence class of valuations on k. A place ν on k is said to be archimedean (resp. non-archimedean) if it consist of archimedean (resp. non-archimedean) valuations. Given a place ν on a global field k we denote by k_ν the completion of k with respect to the metric induced by any of the valuations in ν. The space k_ν is a locally compact field. If ν is a non-archimedean place, the set

$$\mathcal{O}_{k,\nu} = \{\eta \in k_\nu \mid |\eta|_\nu \le 1\}$$

is a subring of k_ν where $|\cdot|_\nu$ is the norm in k_ν induced by ν. We call $\mathcal{O}_{k,\nu}$ the ring of integers of k_ν. For a non-archimedean place ν on k the ring $\mathcal{O}_{k,\nu}$ is an open compact subring of k_ν.

If k is a number field, then non-archimedean places on k are in one-to-one correspondence with prime ideals in $\mathcal{O}_k$ while the archimedean places on k correspond to the finitely many different embeddings of k in $\mathbb{C}$. This is essentially a consequence of Ostrowski's theorem by which any valuation on $\mathbb{Q}$ is equivalent to the p-adic absolute value $|\cdot|_p$ for some prime number p or to the ordinary absolute value induced by the embedding of $\mathbb{Q}$ in $\mathbb{R}$.

Given a collection $\{\Sigma_\lambda\}_{\lambda\in\Lambda}$ of locally compact topological spaces and compact subspaces $\Omega_\lambda \subset \Sigma_\lambda$ for all but finitely many $\lambda \in \Lambda$, there exists a unique topology on the set

$$\prod^{(\text{res})} \Sigma_\lambda = \{(\eta_\lambda) \in \prod \Sigma_\lambda \mid \eta_\lambda \in \Omega_\lambda \text{ for all but finitely many } \lambda \in \Lambda\},$$

for which $\prod \Omega_\lambda$ is a compact subspace. We call the set $\prod^{(res)} \Sigma_\lambda$ together with this topology the *restricted topological product* of $\{\Sigma_\lambda\}_{\lambda\in\Lambda}$ with respect to the subspaces Ω_λ.

Definition 5.1. Let k be a global field and let $\mathcal{P}$ be the collection of places on k. The ring of adèles of k, denoted by $\mathbb{A}_k$, is the topological ring given by the restricted topological product of $\{k_\nu\}_{\nu\in\mathcal{P}}$ with respect to the subspaces $\mathcal{O}_{k,\nu}$. The group of idèles of k, denoted by $\mathbb{I}_k$, is the topological group given by the restricted topological product of $\{k_\nu^*\}_{\nu\in\mathcal{P}}$ with respect to the subspaces $\mathcal{O}_{k,\nu}^*$.

Any global field k can be embedded as a discrete co-compact subfield of the locally compact ring $\mathbb{A}_k$. Likewise k^* embeds diagonally in $\mathbb{I}_k$. The quotient topological group

$$\mathcal{C}_k = \mathbb{I}_k / k^*$$

is called the *idèle class group* of k.

One of the central results in class field theory associates to any abelian extension of a global field k a subgroup of the idèle class group. The main theorem of global class field theory can be stated as follows:

Theorem 5.2. *Let k be a number field and let k^{ab} be its maximal abelian extension. Denote by $\mathcal{D}_k$ the connected component of the identity in $\mathcal{C}_k$. Then there is a canonical isomorphism of topological groups:*

$$\Psi \colon \mathcal{C}_k / \mathcal{D}_k \to \mathrm{Gal}(k^{\mathrm{ab}}/k).$$

In the case of a function field k an analogous result identifies in a canonical way the group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ with the profinite completion of $\mathcal{C}_k$. The isomorphism Ψ is usually referred as the global Artin map or reciprocity map.

5.2 Noncommutative spaces. One of the departure points of noncommutative geometry is the duality between various classes of spaces and algebras of functions canonically associated to these spaces. One classical instance of this duality is furnished by Gelfand's theorem which provides a one-to-one correspondence between compact Hausdorff topological spaces and unital C*-algebras via the functor that sends a space X to its algebra of complex valued continuous functions. Other examples and of this type of duality and refinements thereof abound in the mathematical landscape. In these situations it might be possible to define structures relevant to the study of a space in terms of the corresponding algebras of functions and, in cases in which these definitions do not depend on the commutativity of the underlying algebra, extend them in order to study noncommutative algebras. Geometric information which is difficult to encode with traditional tools might be understood by enlarging the class of spaces in consideration in such a way as to allow the presence of "noncommutative coordinates". In this way we view a noncommutative algebra as defining by duality a noncommutative space for which this algebra plays the role of algebra of coordinates. One may then for example view a noncommutative unital C*-algebra as defining by duality a "noncommutative compact Hausdorff topological space". This process is far from being a mere translation of concepts to another framework even when such a translation might prove to be delicate. Many new phenomena arise in this context and in some situations the classical picture is also enriched (cf. [9]).

5.3 Quantum statistical mechanics. A particular area in which these ideas occur naturally is quantum statistical mechanics. A quantum statistical mechanical system is determined by a C*-algebra $\mathcal{A}$ (the algebra of observables of the system) and a one-parameter group of automorphisms $\sigma_t \in \mathrm{Aut}(\mathcal{A})$, $t \in \mathbb{R}$ (the time evolution of the system). A state on the C*-algebra $\mathcal{A}$ is by definition a norm-one linear functional $\varphi\colon A \to \mathbb{C}$ satisfying the positivity condition $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. In a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ the time evolution σ_t singles out a class of states of the algebra $\mathcal{A}$, the equilibrium states of the system, these are families of states parametrized by a positive real number β corresponding to a thermodynamical parameter (the inverse temperature of the system). The appropriate definition of equilibrium states in the context of quantum statistical mechanics was given by Haag, Hugenholtz and Winnink in [25]. This condition, known after Kubo, Martin and Schwinger as the KMS condition, is given as follows.

Definition 5.3. Let $(\mathcal{A}, \sigma_t)$ be a quantum statistical mechanical system. A state φ on $\mathcal{A}$ satisfies the KMS condition at inverse temperature $0 < \beta < \infty$ if for every $a, b \in \mathcal{A}$ there exists a bounded holomorphic function $F_{a,b}$ on the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \beta\}$, continuous on the closed strip, such that

$$F_{a,b}(t) = \varphi(a\sigma_t(b)), \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a) \quad \text{for all } t \in \mathbb{R}.$$

We call such state a KMS_β state. A KMS_∞ state is by definition a weak limit of KMS_β states as $\beta \to \infty$.

For each $0 < \beta \leq \infty$ the set of KMS_β states associated to the time evolution σ_t is a compact convex space (cf. [6], Section 5.3). We denote by $\mathcal{E}_\beta$ the space of extremal points of the space of KMS_β states. A group $G \subset \mathrm{Aut}(\mathcal{A})$ such that $\sigma_t g = g\sigma_t$ for all $g \in G$ and all $t \in \mathbb{R}$ is called a symmetry group of the system $(\mathcal{A}, \sigma_t)$. If G is a symmetry group of the system $(\mathcal{A}, \sigma_t)$ then G acts on the space of KMS_β states for any β and hence on $\mathcal{E}_\beta$. Inner automorphisms coming from unitaries invariant under the time evolution act trivially on equilibrium states.

Starting with the seminal work of Bost and Connes, [5], various quantum statistical mechanical systems associated to arithmetic data have been studied. Potential applications to the explicit class field theory problem make the understanding of such systems particularly valuable (see [44], [15] and references therein). We recall below the definition and properties of the quantum statistical mechanical system introduced in [5] in the form most adequate for our purposes, which in particular serves as a motivating example for the introduction of endomotives.

5.4 The Bost–Connes system. For a positive integer n consider the cyclic group of order n as a zero-dimensional variety over $\mathbb{Q}$ given by

$$X_n = \mathrm{Spec}(A_n); \quad A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}].$$

We order $\mathbb{N}^\times$ by divisibility. For $n|m$ the canonical morphism $X_m \to X_n$ is a morphism in $\mathcal{V}^0(\mathbb{Q})$ and we can view $\{X_n\}_{n \in \mathbb{N}^\times}$ as a projective system of zero-dimensional algebraic varieties over $\mathbb{Q}$. The profinite limit

$$X = \varprojlim_n X_n$$

corresponds to the direct limit of finite dimensional algebras:

$$A = \varinjlim_n A_n = \varinjlim_n \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}] \simeq \mathbb{Q}[\mathbb{Q}/\mathbb{Z}].$$

Denote by $\{e(r) \mid r \in \mathbb{Q}/\mathbb{Z}\}$ the canonical basis of A. The multiplicative abelian semigroup $\mathbb{N}^\times$ acts as a semigroup of endomorphisms of the algebra A by

$$\rho_n \colon A \to A, \quad e(r) \mapsto \tfrac{1}{n} \sum_{l \in \mathbb{Q}/\mathbb{Z}, ln=r} e(l).$$

The algebra of continuous functions on the profinite space $X(\bar{\mathbb{Q}})$ coincides with $C^*(\mathbb{Q}/\mathbb{Z})$, the group C*-algebra of $\mathbb{Q}/\mathbb{Z}$ (i.e., the completion of $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ in its regular representation). The semigroup action of $\mathbb{N}^\times$ on A extends to an action of semigroup action of $\mathbb{N}^\times$ on $C^*(\mathbb{Q}/\mathbb{Z})$ and the semigroup crossed product algebra $\mathcal{A}_\mathbb{Q} = A \rtimes \mathbb{N}^\times$ is a rational sub-algebra of the semigroup crossed product C*-algebra

$$\mathcal{A} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times.$$

The C*-algebra $\mathcal{A}$ coincides with the C*-algebra generated by elements $e(r)$, $r \in \mathbb{Q}/\mathbb{Z}$ (corresponding the generators of the group algebra of $\mathbb{Q}/\mathbb{Z}$) and $\mu_n, n \in \mathbb{N}^\times$ (corresponding the generators of the semigroup action) and satisfying the relations:

$$e(0) = 1, \quad e(r)e(s) = e(r+s), \quad e(r)^* = e(-r) \quad \text{for all } r, s \in \mathbb{Q}/\mathbb{Z},$$
$$\mu_n \mu_k = \mu_{nk}, \quad \mu_n^* \mu_n = 1 \quad \text{for all } n, k \in \mathbb{N}^\times,$$
$$\mu_n e(r) \mu_n^* = \rho_n(e(r)) \quad \text{for all } n \in \mathbb{N}^\times,\ r \in \mathbb{Q}/\mathbb{Z}.$$

It is possible to define a time evolution σ_t on the C*-algebra $\mathcal{A}$ by taking

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it}.$$

Each element $\eta \in \hat{\mathbb{Z}}^* \simeq \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ defines a representation π_η of the C*-algebra $\mathcal{A}$ as an algebra of operators on the Hilbert space $l^2(\mathbb{N}^\times)$ via

$$\pi_\eta(e(r))\epsilon_k = e^{2\pi \imath \eta r} \varepsilon_n, \quad \pi_\eta(\mu_n)\varepsilon_k = \varepsilon_{nk},$$

where $\{\varepsilon_k\}_{k \in \mathbb{N}^\times}$ is the canonical basis for $l^2(\mathbb{N}^\times)$. For any of these representations the time evolution σ_t can be implemented via the Hamiltonian $H(\varepsilon_k) = (\log k)\epsilon_k$ as

$$\pi_\eta(\sigma_t(a)) = e^{\imath t H} \pi_\eta(a) e^{-\imath t H}.$$

The partition function of the system, defined as $\mathrm{Trace}(e^{\beta H})$ is then given by the Riemann zeta function $\zeta(\beta)$. The structure of equilibrium states for this system was studied in [5], where it is shown that for $0 < \beta \leq 1$ there exist a unique KMS_β state while for any $1 < \beta \leq \infty$ there are infinitely many states and the space of extremal states $\mathcal{E}_\beta$ is homeomorphic to $\hat{\mathbb{Z}}^*$. The group $\hat{\mathbb{Z}}^*$ acts as a group of symmetries of the system $(\mathcal{A}, \sigma_t)$ and the induced action on $\mathcal{E}_\beta$ for $1 < \beta \leq \infty$ is free and transitive. The arithmetic of abelian extensions of $\mathbb{Q}$ is encoded in this system in the following way:

Theorem 5.4 (Bost, Connes). *For every $\varphi \in \mathcal{E}_\infty$ and every $a \in \mathcal{A}_{\mathbb{Q}}$ the value $\varphi(a)$ is algebraic over $\mathbb{Q}$. Moreover $\mathbb{Q}^{\mathrm{ab}}$ is generated by values of this form and for all $\varphi \in \mathcal{E}_\infty$, $\gamma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}|\mathbb{Q})$ and $a \in \mathcal{A}_{\mathbb{Q}}$ one has*

$$\gamma\varphi(a) = \varphi(\Psi^{-1}(\gamma)a),$$

where $\Psi\colon \mathcal{C}_{\mathbb{Q}}/D_{\mathbb{Q}} = \widehat{\mathbb{Z}}^* \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}|\mathbb{Q})$ is the class field theory isomorphism.

5.5 Algebraic and analytic endomotives. In this section we briefly survey the theory of endomotives as introduced in [11]. It provides a systematic way to construct associative algebras analogous to the rational sub-algebra of the Bost–Connes system in terms of arithmetic data. The resulting category provides an enlargement of the category of Artin motives over a number field to a category of arithmetic noncommutative spaces.

Let k be a number field. Given a projective system $\{X_i\}_{i\in I}$ of varieties in $\mathcal{V}^0(k)$, I a countable partially ordered set, we can consider the direct limit of algebras

$$A = \varinjlim_i A_i,$$

where $X_i = \operatorname{Spec} A_i$. Assume that S is an abelian semigroup acting by algebra endomorphisms on A such that for any $s \in S$ the corresponding endomorphism ρ_s induces an isomorphism

$$A \simeq p_s A,$$

where $p_s = p_s^2 = \rho_s(1)$.

Definition 5.5. An algebraic endomotive over k is given by an associative algebra of the form

$$\mathcal{A}_k = A \rtimes S$$

corresponding a projective system $\{X_i\}_{i\in I}$ of varieties in $\mathcal{V}^0(k)$ and a semigroup S as above.

Fix an algebraic closure $\bar{k}$ of k. For an algebraic endomotive $\mathcal{A}_k$ with corresponding projective system $\{X_i\}_{i\in I}$ the set of algebraic points of the pro-variety $X = \varprojlim_i X_i$ is given by the profinite (compact Hausdorff) space:

$$X(\bar{k}) = \mathrm{Hom}_{k\text{-alg}}(A, \bar{k}).$$

Given an embedding $\bar{k} \hookrightarrow \mathbb{C}$ we can identify A with a sub-algebra of $C(X(\bar{k}))$. Pure states in $C(X(\bar{k}))$ attain algebraic values when restricted to A. Moreover, the natural action of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ on $X(\bar{k})$ induces an action of $\mathrm{Gal}(\bar{k}/k)$ on the C*-algebra

$$\mathcal{A} = C(X(\bar{k})) \rtimes S,$$

in which the algebraic endomotive $\mathcal{A}_k$ embeds as a rational sub-algebra. An analytic endomotive is by definition a C*-algebra of the above form. In the case of an endomotive given by abelian extensions of k the corresponding action factors through $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ and the following result holds:

Theorem 5.6 (Connes, Consani, Marcolli). *Let $\mathcal{A}_k$ be an algebraic endomotive over k with corresponding projective system of varieties $\{X_i\}_{i\in I}$ and semigroup S. Assume that for each $i \in I$ the algebra A_i is a finite product of normal abelian field extensions of k. Then the algebras A and $\mathcal{A}_k$ are globally invariant under the action of* $\mathrm{Gal}(\bar{k}/k)$ *on $\mathcal{A} = C(X(\bar{k})) \rtimes S$. Moreover, any state φ on $\mathcal{A}$ induced by a pure state on $C(X(\bar{k}))$ satisfies*

$$\gamma\varphi(a) = \varphi(\gamma^{-1}a)$$

for all $a \in \mathcal{A}_k$ and all $\gamma \in$ $\mathrm{Gal}(k^{\mathrm{ab}}/k)$.

Example 5.7. Let (Y, y_0) be a pointed algebraic variety over k and let S be an abelian semigroup of finite morphisms from Y to itself. Assume moreover that any morphism in S fixes y_0 and is unramified over y_0. For any $s \in S$ let

$$X_s = \{y \in Y \mid s(y) = y_0\}.$$

Order S by divisibility, then the system $\{X_s\}_{s\in S}$ defines an algebraic endomotive over k with S as semigroup of endomorphisms.

Example 5.8. Let $\mathbb{G}_m$ denote the multiplicative group viewed as a variety over $\mathbb{Q}$. The power maps $z \mapsto z^n$ for $n \in \mathbb{N}^\times$ define finite self-morphisms on $\mathbb{G}_m$ fixing the point $1 \in \mathbb{G}_m$ and unramified over it. The algebraic endomotive associated to $(\mathbb{G}_m, 1)$ and $S = \mathbb{N}^\times$ as in Example 5.7 is the arithmetic subalgebra $\mathcal{A}_\mathbb{Q}$ of the Bost–Connes system.

Example 5.9. Let E be an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field K. Any endomorphism of E in $\mathcal{O}^\times$ fixes 0 and is unramified over it. The algebraic endomotive associated to $(E, 0)$ and $S = \mathcal{O}^\times$ as in Example 5.7 is the arithmetic subalgebra of the quantum statistical mechanical system considered in [16]. This system encodes the class field theory of the field K.

An algebraic endomotive over k defines a groupoid in a natural way. By considering groupoid actions satisfying a suitable étale condition it is possible to extend morphisms in $\mathbf{Corr}^0(k, F)$ in order to define correspondences between algebraic endomotives. The pseudo-abelian envelope of the category so obtained is the category of algebraic endomotives $\mathbf{EndMot}^0(k, F)$. By construction $\mathbf{Mot}^0(k, F)$ embeds as a full subcategory of $\mathbf{EndMot}^0(k, F)$.

The groupoid picture likewise allows to define a category of analytic endomotives $\mathbf{C^*EndMot}^0(k, F)$ as the pseudo-abelian envelope of the category of correspondences between analytic endomotives. The map that assigns to an algebraic endomotive over k with projective system $\{X_i\}_{i\in I}$ and semigroup S the analytic endomotive $C(X(\bar{k})) \rtimes S$, $X = \varprojlim_i X_i$, extends to a tensor functor

$$\mathbf{EndMot}^0(k, F) \to \mathbf{C^*EndMot}^0(k, F)$$

on which the Galois group $\mathrm{Gal}(\bar{k}/k)$ acts by natural transformations.

Motivic ideas in noncommutative geometry

An appendix by Matilde Marcolli

There has been in recent years a very fruitful interplay between ideas originally developed in the context of Grothendieck's theory of motives of algebraic varieties and techniques and notions arising in the context of noncommutative geometry.

Two main directions have become prominent: one based on treating noncommutative spaces as algebras, and importing motivic ideas by extending the notion of morphisms of noncommutative spaces to include Morita equivalences through correspondences realized by bimodules and other types of morphisms in larger categories like cyclic modules. This allows for cohomological methods based on cyclic (co)homology to be employed in a setting that provides an analog of the motivic ideas underlying the Weil proof of the Riemann hypothesis for function fields. It is this approach, developed in [11], [12], [15], that we focus on mostly in this survey. It concentrates on a category of noncommutative motives that are built out of the simplest class of motives of algebraic varieties, the Artin motives, which are motives of zero-dimensional algebraic varieties.

At the same time, there is a more general and very broad approach to motives in the noncommutative geometry setting, developed by Kaledin, Kontsevich, Tabuada, and others [33], [34], [51], based on the idea of representing noncommutative spaces as categories instead of algebras, and the related circle of ideas of derived algebraic geometry, [35], see also the short survey [41]. As we argue briefly in §4 below, one can expect that a merging of these two approaches will lead to some very interesting generalizations of some of the results that we review here.

1 Noncommutative motives and cyclic cohomology

In noncommutative geometry one encounters a problem that is very familiar to the context of algebraic geometry. Namely, if one thinks of noncommutative spaces as being described by associative algebras, then the category of algebras over a field with algebra homomorphisms is not abelian or even additive. Moreover, it is well known that morphisms of algebras are too restrictive a notion of morphisms for noncommutative spaces, as they do not account for the well known phenomenon of Morita equivalence.

Thus, one needs to embed the category of associative algebras with algebra homomorphisms inside an abelian (or at least pseudo-abelian) category with a larger collection of morphisms that include the correspondences given by tensoring with bimodules, as in the case of Morita equivalences. The objects of a category with these properties can be regarded as "noncommutative motives", in the same sense as the motives of algebraic varieties are the objects of an abelian category (or pseudo-abelian, or triangulated in the mixed case) that contains the category of algebraic varieties.

A first construction of such a category of noncommutative motives was obtained in [8], using the notion of cyclic modules.

One first defines the *cyclic category* Λ as the category that has an object $[n]$ for each positive integer and has morphisms generated by the morphisms $\delta_i : [n-1] \to [n]$, $\sigma_j : [n+1] \to [n]$, and $\tau_n : [n] \to [n]$, with the relations

$$
\begin{aligned}
\delta_j \delta_i &= \delta_i \delta_{j-1} \quad \text{for } i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \quad \text{for } i \le j, \\
\sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j, \\ 1_n & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1} \sigma_j & \text{if } i > j+1, \end{cases} \\
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \quad \text{for } 1 \le i \le n, \\
\tau_n \delta_0 &= \delta_n, \\
\tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} \quad \text{for } 1 \le i \le n, \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2 \\
\tau_n^{n+1} &= 1_n.
\end{aligned}
$$

Given a category $\mathcal{C}$, one defines as *cyclic objects* the covariant functors $\Lambda \to \mathcal{C}$. In particular, we consider the case where $\mathcal{C} = \mathrm{Vect}_{\mathbb{K}}$ is the category of vector spaces over a field $\mathbb{K}$ and we refer to the cyclic objects as *cyclic modules*, or $\mathbb{K}(\Lambda)$-modules.

In particular, consider a unital associative algebra $\mathcal{A}$ over a field $\mathbb{K}$. One associates to $\mathcal{A}$ a $\mathbb{K}(\Lambda)$-module $\mathcal{A}^\natural$, which is the covariant functor $\Lambda \to \mathrm{Vect}_{\mathbb{K}}$ that assigns to objects in Λ the vector spaces

$$[n] \Rightarrow \mathcal{A}^{\otimes^{(n+1)}} = \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$$

and to the generators of the morphisms of Λ the linear maps

$$
\begin{aligned}
\delta_i &\Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^n), \\
\sigma_j &\Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^0 \otimes \cdots \otimes a^i \otimes 1 \otimes a^{i+1} \otimes \cdots \otimes a^n), \\
\tau_n &\Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^n \otimes a^0 \otimes \cdots \otimes a^{n-1}).
\end{aligned}
$$

The category of cyclic modules is an abelian category, and the construction above shows that one can embed inside it a copy of the category of associative algebras, hence the cyclic modules can be regarded as noncommutative motives. Notice that there are many more objects in the category of cyclic modules than those that come from associative algebras. For example, being an abelian category, kernels and cokernels of morphisms of cyclic modules are still cyclic modules even when, for instance, one does not have cokernels in the category of algebras.

Moreover, one sees that there are many more morphisms between cyclic modules than between algebras and that the type of morphisms one would like to have between noncommutative spaces, such as Morita equivalences, are included among the morphisms of cyclic modules.

In particular, all the following types of morphisms exist in the category of cyclic modules.

- Morphisms of algebras $\phi\colon \mathcal{A} \to \mathcal{B}$ induce morphisms of cyclic modules $\phi^\natural\colon \mathcal{A}^\natural \to \mathcal{B}^\natural$.
- Traces $\tau\colon \mathcal{A} \to \mathbb{K}$ induce morphisms of cyclic modules $\tau^\natural\colon \mathcal{A}^\natural \to \mathbb{K}^\natural$ by setting $\tau^\natural(x^0 \otimes \cdots \otimes x^n) = \tau(x^0 \dots x^n)$.
- $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{E}$ induce morphisms of cyclic modules, $\mathcal{E}^\natural = \tau^\natural \circ \rho^\natural$, by composing $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ and $\tau\colon \mathrm{End}_{\mathcal{B}}(\mathcal{E}) \to \mathcal{B}$.

Moreover, it was shown in [8] that in the abelian category of cyclic modules the Ext functors recover cyclic cohomology of algebras by

$$\mathrm{HC}^n(\mathcal{A}) = \mathrm{Ext}^n(\mathcal{A}^\natural, \mathbb{K}^\natural).$$

2 Artin motives and the category of endomotives

The simplest category of motives of algebraic varieties is the category of Artin motives, which corresponds to zero-dimensional varieties over a field $\mathbb{K}$ (which we take here to be a number field), with correspondences that are given by formal linear combinations of subvarieties $Z \subset X \times Y$ in the product. Because everything is zero-dimensional, in this case one does not have to worry about the different equivalence relations on cycles. Artin motives over $\mathbb{K}$ not only form an abelian category, but in fact a Tannakian category with motivic Galois group given by the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$.

The category of endomotives was introduced in [11] as a category of noncommutative spaces that are built out of towers of Artin motives with endomorphisms actions.

At the algebraic level, one considers as objects crossed product algebras $\mathcal{A}_{\mathbb{K}} = A \rtimes S$, where A is a commutative algebra over $\mathbb{K}$, obtained as a direct limit $A = \varinjlim_\alpha A_\alpha$ of finite dimensional reduced algebras over $\mathbb{K}$, which correspond under $X_\alpha = \mathrm{Spec}(A_\alpha)$ to zero-dimensional algebraic varieties X_α, Artin motives over $\mathbb{K}$. The direct limit A of algebras corresponds to a pro-variety $X = \varprojlim_\alpha X_\alpha$.

The datum S is a unital abelian semigroup, which acts on A by endomorphisms with $\rho\colon A \xrightarrow{\simeq} eAe$, where $e = \rho(1)$ is an idempotent $e^2 = e$.

Morphisms between these objects are also constructed out of morphisms (correspondences) in the category of Artin motives, via projective limits, compatibly with the semigroup actions. More precisely, if $\mathcal{G}(X_\alpha, S)$ denotes the groupoid of the action of S on X, the morphisms of endomotives are given by étale correspondences, which are formal linear combinations of $\mathcal{G}(X_\alpha, S) - \mathcal{G}(X'_{\alpha'}, S')$ spaces Z for which the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale. This means that, when representing $Z = \mathrm{Spec}(M)$, with

M an $\mathcal{A}_{\mathbb{K}}$–$\mathcal{A}'_{\mathbb{K}}$ bimodule, M is finite projective as a right $\mathcal{A}_{\mathbb{K}}$-module. Morphisms are then given by the $\mathbb{Q}$-linear space $M((X_\alpha, S), (X'_{\alpha'}, S'))$ formal linear combinations $U = \sum_i a_i Z_i$ of étale correspondences as above. The composition of morphisms is then given by the fibered product $Z \circ W = Z \times_{\mathcal{G}'} W$ over the groupoid of the action of S' on X'.

At the analytic level, one considers the topological space $\mathcal{X} = X(\bar{\mathbb{K}})$ with the action of the semigroup S. The topology is the one of the projective limit, which makes $X(\bar{\mathbb{K}})$ into a totally disconnected (Cantor-like) compact Hausdorff space. One can therefore consider the crossed product C*-algebras $\mathcal{A} = C(X(\bar{\mathbb{K}})) \rtimes S = C^*(\mathcal{G})$. One imposes a uniform condition, which provides a probability measure $\mu = \varprojlim \mu_\alpha$ obtained using the counting measures on the X_α, with $\frac{d\rho^*\mu}{d\mu}$ locally constant on $X(\bar{\mathbb{K}})$. Integration with respect to this measure gives a state φ on the C*-algebra $\mathcal{A}$.

One can also extend morphisms given by étale correspondences to this analytic setting. These are given by spaces $\mathcal{Z}$ with maps $g\colon \mathcal{Z} \to \mathcal{X}$ with discrete fiber and such that 1 is a compact operator on the right module $\mathcal{M}_{\mathcal{Z}}$ over $C(\mathcal{X})$ from the $C_c(\mathcal{G})$-valued inner product $\langle \xi, \eta\rangle(x, s) := \sum_{z\in g^{-1}(x)} \bar{\xi}(z)\eta(z \circ s)$.

For $\mathcal{G}$-$\mathcal{G}'$ spaces defining morphisms of algebraic endomotives, one can consider $Z \mapsto Z(\bar{\mathbb{K}}) = \mathcal{Z}$ and obtain a correspondence of analytic endomotives, with $C_c(\mathcal{Z})$ a right module over $C_c(\mathcal{G})$. These morphisms induce morphisms in the KK category and in the category of cyclic modules.

The analytic endomotive $\mathcal{A}$ is endowed with a Galois action of $G = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$, as an action on the characters $X(\bar{\mathbb{K}}) = \mathrm{Hom}(A, \bar{\mathbb{K}})$ by

$$A \xrightarrow{\chi} \bar{\mathbb{K}} \mapsto A \xrightarrow{\chi} \bar{\mathbb{K}} \xrightarrow{g} \bar{\mathbb{K}}.$$

The action is also compatible with the endomorphisms action of S since the latter acts on the characters by pre-composition. Thus, the Galois group acts by automorphisms of the crossed product algebra $\mathcal{A} = C(\mathcal{X}) \rtimes S$.

2.1 The Bost–Connes endomotive. A prototype example of an endomotive is the noncommutative space associated to the quantum statistical mechanical system constructed by Bost and Connes in [5]. A more transparent geometric interpretation of this noncommutative space as the moduli space of 1-dimensional $\mathbb{Q}$-lattices up to scale, modulo the equivalence relation of commensurability, was given in [14].

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ and a (possibly degenerate) labeling of its torsion points via a group homomorphism

$$\phi\colon \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda.$$

In the special case where ϕ is an isomorphism one says that the $\mathbb{Q}$-lattice is invertible.

The equivalence relation of commensurability is defined by setting $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ whenever $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$.

The quotient of the space of $\mathbb{Q}$-lattices by the commensurability relation is a "bad quotient" in the sense of ordinary geometry, but it can be described by a noncommutative space whose algebra of functions is the convolution algebra of the equivalence relation.

In the 1-dimensional case a $\mathbb{Q}$-lattices is specified by the data

$$(\Lambda, \phi) = (\lambda\mathbb{Z}, \lambda\rho)$$

for some $\lambda > 0$ and some $\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$. If we consider the lattices up to scaling, we eliminate the factor λ and we are left with a space whose algebra of functions is $C(\hat{\mathbb{Z}})$.

The commensurability relation is then expressed by the action of the semigroup $\mathbb{N} = \mathbb{Z}_{>0}$ which maps $\alpha_n(f)(\rho) = f(n^{-1}\rho)$ when one can divide by n and sets the result to zero otherwise.

The quotient of the space of 1-dimensional $\mathbb{Q}$-lattices up to scale by commensurability is then realized as a noncommutative space by the crossed product algebra $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$. This can also be written as a convolution algebra for a partially defined action of $\mathbb{Q}^*_+$, with

$$f_1 * f_2(r, \rho) = \sum_{s \in \mathbb{Q}^*_+, s\rho \in \hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s, \rho)$$

with adjoint $f^*(r, \rho) = \overline{f(r^{-1}, r\rho)}$. This is the algebra of the groupoid of the commensurability relation. It is isomorphic to the Bost–Connes (BC) algebra of [5].

As an algebra over $\mathbb{Q}$, it is given by $\mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$, and it has an explicit presentation by generators and relations of the form

$$\begin{aligned}
\mu_n \mu_m &= \mu_{nm}, \\
\mu_n \mu_m^* &= \mu_m^* \mu_n \quad \text{when } (n, m) = 1, \\
\mu_n^* \mu_n &= 1, \\
e(r + s) &= e(r)e(s), \quad e(0) = 1, \\
\rho_n(e(r)) &= \mu_n e(r) \mu_n^* = \tfrac{1}{n} \sum_{ns=r} e(s).
\end{aligned}$$

The C*-algebra is then given by $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$, where one uses the identification, via Pontrjagin duality, between $C(\hat{\mathbb{Z}})$ and $C^*(\mathbb{Q}/\mathbb{Z})$.

The time evolution of the BC quantum statistical mechanical system is given in terms of generators and relations by

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it} \mu_n$$

and it is generated by a Hamiltonian $H = \frac{d}{dt}\sigma_t|_{t=0}$ with partition function $\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$, in the representations on the Hilbert space $\ell^2(\mathbb{N})$ parameterized by the invertible $\mathbb{Q}$-lattices $\rho \in \hat{\mathbb{Z}}^*$. These representation π_ρ on $\ell^2(N)$, for $\rho \in \hat{\mathbb{Z}}^*$, are given on generators by

$$\mu_n \epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m,$$

where $\zeta_r = \rho(e(r))$ is a root of unit.

Given a C*-algebra with time evolution, one can consider states, that is, linear functionals $\varphi\colon \mathcal{A} \to \mathbb{C}$, with $\varphi(1) = 1$ and $\varphi(a^*a) \geq 0$, that are equilibrium states for the time evolution. As a function of a thermodynamic parameter (an inverse temperature β), these are specified by the KMS condition: $\varphi \in \mathrm{KMS}_\beta$ for some $0 < \beta < \infty$ if for all $a, b \in \mathcal{A}$ there exists a holomorphic function $F_{a,b}(z)$ on the strip $I_\beta = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \beta\}$, continuous on the boundary ∂I_β, and such that, for all $t \in \mathbb{R}$,

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

In the case of the BC system the KMS states are classified in [5]: the low temperature extremal KMS states, for $\beta > 1$ are of the form

$$\varphi_{\beta,\rho}(a) = \frac{\mathrm{Tr}(\pi_\rho(a)e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*,$$

while at higher temperatures there is a unique KMS state.

At zero temperature the evaluations $\varphi_{\infty,\rho}(e(r)) = \zeta_r$, which come from the projection on the kernel of the Hamiltonian,

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a)\epsilon_1 \rangle,$$

exhibit an intertwining of Galois action on the values of states on the arithmetic subalgebra and symmetries of the quantum statistical mechanical system: for $a \in \mathcal{A}_{\mathbb{Q},BC}$ and $\gamma \in \hat{\mathbb{Z}}^*$, one has

$$\varphi_{\infty,\rho}(\gamma a) = \theta_\gamma(\varphi_{\infty,\rho}(a)),$$

where

$$\theta\colon \hat{\mathbb{Z}}^* \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$$

is the class field theory isomorphism.

The BC algebra is an endomotive with $A = \varinjlim_n A_n$, for $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ and the abelian semigroup action of $S = \mathbb{N}$ on $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.

A more general class of endomotives was constructed in [11] using self-maps of algebraic varieties. One constructs a system (A, S) from a collection S of self-maps of algebraic varieties $s\colon Y \to Y$ and their iterations, with $s(y_0) = y_0$ unbranched and s of finite degree, by setting $X_s = s^{-1}(y_0)$ and taking the projective limit $X = \varprojlim X_s = \mathrm{Spec}(A)$ under the maps

$$\xi_{s,s'}\colon X_{s'} \to X_s, \quad \xi_{s,s'}(y) = r(y), \quad s' = rs \in S.$$

The BC endomotive is a special case in this class, with $Y = \mathbb{G}_m$ with self-maps $u \mapsto u^k$

$$s_k\colon P(t, t^{-1}) \mapsto P(t^k, t^{-k}), \quad k \in \mathbb{N}, \; P \in \mathbb{Q}[t, t^{-1}],$$
$$\xi_{k,\ell}(u(\ell)) = u(\ell)^{k/\ell}, \quad u(\ell) = t \mod t^\ell - 1.$$

One then has $X_k = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}]/(t^k - 1)) = s_k^{-1}(1)$ and $X = \varprojlim_k X_k$ with

$$u(\ell) \mapsto e(1/\ell) \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}].$$

One can identify the algebras $C(X(\bar{\mathbb{Q}})) = C(\hat{\mathbb{Z}})$.

2.2 Time evolution and KMS states. An object (X, S) in the category of endomotives, constructed as above, determines the following data:

- a C*-algebra $\mathcal{A} = C(\mathcal{X}) \rtimes S$;
- an arithmetic subalgebra $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ defined over $\mathbb{K}$;
- a state $\varphi \colon \mathcal{A} \to \mathbb{C}$ from the uniform measure on the projective limit;
- an action of the Galois group by automorphisms $G \subset \mathrm{Aut}(\mathcal{A})$.

As shown in [11], see also §4 of [15], these data suffice to apply the thermodynamic formalism of quantum statistical mechanics. In fact, Tomita–Takesaki theory shows that one obtains from the state φ a time evolution, for which φ is a KMS_1 state.

One starts with the GNS representation $\mathcal{H}_\varphi$. The presence of a cyclic and separating vector ξ for this representation, so that $\mathcal{M}\xi$ and $\mathcal{M}'\xi$ are both dense in $\mathcal{H}_\varphi$, with $\mathcal{M}$ the von Neumann algebra generated by $\mathcal{A}$ in the representation, ensures that one has a densely defined operator

$$\begin{aligned} S_\varphi &\colon \mathcal{M}\xi \to \mathcal{M}\xi, \quad a\xi \mapsto S_\varphi(a\xi) = a^*\xi, \\ S_\varphi^* &\colon \mathcal{M}'\xi \to \mathcal{M}'\xi, \quad a'\xi \mapsto S_\varphi^*(a'\xi) = a'^*\xi, \end{aligned}$$

which is closable and has a polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ with J_φ a conjugate-linear involution $J_\varphi = J_\varphi^* = J_\varphi^{-1}$ and $\Delta_\varphi = S_\varphi^* S_\varphi$ a self-adjoint positive operator with $J_\varphi \Delta_\varphi J_\varphi = S_\varphi S_\varphi^* = \Delta_\varphi^{-1}$.

Tomita–Takesaki theory then shows that $J_\varphi \mathcal{M} J_\varphi = \mathcal{M}'$ and $\Delta_\varphi^{-it} \mathcal{M} \Delta_\varphi^{it} = \mathcal{M}$, so that one obtains a time evolution (the modular automorphism group)

$$\sigma_t(a) = \Delta_\varphi^{-it} a \Delta_\varphi^{it} \quad a \in \mathcal{M},$$

for which the state φ is a KMS_1 state.

2.3 The classical points of a noncommutative space. Noncommutative spaces typically do not have points in the usual sense of characters of the algebra, since noncommutative algebras tend to have very few two-sided ideals. A good way to replace characters as a notion of points on a noncommutative space is by using extremal states, which in the commutative case correspond to extremal measures supported on points. While considering all states need not lead to a good topology on this space of points, in the presence of a natural time evolution, one can look at only those states that are equilibrium states for the time evolution. The notion of KMS states provides equilibrium states at a fixed temperature, or inverse temperature β. The extremal KMS_β states thus give a good working notion of points on a noncommutative space, with the interesting phenomenon that the set of points becomes temperature dependent and subject to phase transitions at certain critical temperatures.

In particular one can consider, depending on the inverse temperature β, that subset Ω_β of the extremal KMS states that are of Gibbs form, namely that are obtained from type I_∞ factor representations. In typical cases, these arise as low temperature KMS-states, below a certain critical temperature and are then stable when going to lower temperatures, so that one has injective maps $c_{\beta',\beta}\colon \Omega_\beta \to \Omega_{\beta'}$ for $\beta' > \beta$.

For a state $\epsilon \in \Omega_\beta$ one has an irreducible representation $\pi_\epsilon\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}(\epsilon))$, where the Hilbert space of the GNS representation decomposes as $\mathcal{H}_\epsilon = \mathcal{H}(\epsilon) \otimes \mathcal{H}'$ with $\mathcal{M} = \{T \otimes 1 \mid T \in \mathcal{B}(\mathcal{H}(\epsilon))\}$. The time evolution in this representation is generated by a Hamiltonian $\sigma_t^\varphi(\pi_\epsilon(a)) = e^{itH}\pi_\epsilon(a)e^{-itH}$ with $\mathrm{Tr}(e^{-\beta H}) < \infty$, so that the state can be written in Gibbs form

$$\epsilon(a) = \frac{\mathrm{Tr}(\pi_\epsilon(a)e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}.$$

The Hamiltonian H is not uniquely determined, but only up to constant shifts $H \leftrightarrow H + c$ so that one obtains a real line bundle $\widetilde{\Omega}_\beta = \{(\varepsilon, H)\}$, with $\lambda\,(\varepsilon, H) = (\varepsilon, H + \log\lambda)$ for $\lambda \in \mathbb{R}_+^*$. The fibration $\mathbb{R}_+^* \to \widetilde{\Omega}_\beta \to \Omega_\beta$ has a section $\mathrm{Tr}(e^{-\beta\, H}) = 1$, so it can be trivialized as $\widetilde{\Omega}_\beta \simeq \Omega_\beta \times \mathbb{R}_+^*$.

Besides equilibrium KMS states, an algebra with a time evolution also gives rise to a *dual system* $(\hat{\mathcal{A}}, \theta)$, which is the algebra obtained by taking the crossed product with the time evolution, endowed with a scaling action by the dual group.

Namely, one considers the algebra $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ given by functions $x, y \in \mathcal{S}(\mathbb{R}, \mathcal{A}_\mathbb{C})$ with convolution product $(x \star y)(s) = \int_\mathbb{R} x(t)\,\sigma_t(y(s-t))\,dt$. One equivalently writes elements of $\hat{\mathcal{A}}$ formally as $\int x(t)U_t\,dt$, where U_t are the unitaries that implement the $\mathbb{R}$ action σ_t.

The scaling action θ of $\lambda \in \mathbb{R}_+^*$ on $\hat{\mathcal{A}}$ is given by

$$\theta_\lambda\Big(\int x(t)U_t\,dt\Big) = \int \lambda^{it} x(t)U_t\,dt.$$

A point $(\varepsilon, H) \in \widetilde{\Omega}_\beta$ determines an irreducible representation of $\hat{\mathcal{A}}$ by setting

$$\pi_{\varepsilon,H}\Big(\int x(t)U_t\,dt\Big) = \int \pi_\varepsilon(x(t))e^{itH}\,dt,$$

compatibly with the scaling action: $\pi_{\varepsilon,H} \circ \theta_\lambda = \pi_{\lambda(\varepsilon,H)}$.

When restricting to those elements $x \in \hat{\mathcal{A}}_\beta \subset \hat{\mathcal{A}}$ that have analytic continuation to strip of KMS_β with rapid decay along the boundary, one obtains trace class operators [11]

$$\pi_{\varepsilon,H}\Big(\int x(t)\,U_t\,dt\Big) \in \mathcal{L}^1(\mathcal{H}(\varepsilon)).$$

2.4 Restriction as a morphism of noncommutative motives. It is shown in [11] that one can define a restriction map from a noncommutative space to its classical points, where the latter are defined, as above, in terms of the low temperature extremal KMS states.

This restriction map does not exist as a morphism of algebras, but it does exist as a morphism in an abelian category of noncommutative motives that contains the category of algebras, namely the category of cyclic modules described above.

In fact, one can use the representations $\pi(x)(\varepsilon, H)$ and the trace class property to obtain a map

$$\hat{\mathcal{A}}_\beta \xrightarrow{\pi} C(\widetilde{\Omega}_\beta, \mathcal{L}^1) \xrightarrow{\mathrm{Tr}} C(\widetilde{\Omega}_\beta),$$
$$\pi(x)(\varepsilon, H) = \pi_{\varepsilon,H}(x) \text{ for all } (\varepsilon, H) \in \widetilde{\Omega}_\beta,$$

under a technical hypothesis on the vanishing of obstructions; see [11] and §4 of [15]. Because this map involves taking a trace, it is not a morphism in the category of algebras. However, as we have discussed above, traces are morphisms in the category of cyclic modules, so one regards the above map as a map of the corresponding cyclic modules,

$$\hat{\mathcal{A}}_\beta^\natural \xrightarrow{\pi} C(\widetilde{\Omega}_\beta, \mathcal{L}^1)^\natural, \quad \hat{\mathcal{A}}_\beta^\natural \xrightarrow{\delta=(\mathrm{Tr}\circ\pi)^\natural} C(\tilde{\Omega}_\beta)^\natural.$$

This is equivariant for the scaling action of $\mathbb{R}_+^*$.

Moreover, we know by [8] that the category of cyclic modules is an abelian category. This means that the cokernel of this restriction map exists as a cyclic module, even though it does not come from an algebra. In [11] we denoted this cokernel as $D(\mathcal{A}, \varphi) = \mathrm{Coker}(\delta)$. One can compute its cyclic homology $\mathrm{HC}_0(D(\mathcal{A}, \varphi))$, which also has an induced scaling action of $\mathbb{R}_+^*$, as well as an induced representation of the Galois group G, coming from the Galois representation on the endomotive $\mathcal{A}$.

This gives a space (not a noncommutative space but a noncommutative motive) $D(\mathcal{A}, \varphi)$ whose cohomology $\mathrm{HC}_0(D(\mathcal{A}, \varphi))$ is endowed with a scaling and a Galois action. These data provide an analog in the noncommutative setting of the Frobenius action on étale cohomology in the context of motives of algebraic varieties.

2.5 The Bost–Connes endomotive and the adèles class space. In [11] and [12] the motivic setting described above was applied in particular to the case of the Bost–Connes endomotive $\mathcal{A} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ with the state given by the measure $\varphi(f) = \int_{\hat{\mathbb{Z}}} f(1, \rho)\, d\mu(\rho)$ and the resulting time evolution recovering the original time evolution of the Bost–Connes quantum statistical mechanical system, $\sigma_t(f)(r, \rho) = r^{it} f(r, \rho)$. In this case then the space of classical points is given by

$$\tilde{\Omega}_\beta = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^* = C_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$$

for small temperatures $\beta > 1$.

The dual system is the groupoid algebra of the commensurability relation on $\mathbb{Q}$-lattices, considered not up to scaling, $\hat{\mathcal{A}} = C^*(\tilde{\mathcal{G}})$, where one identifies

$$h(r, \rho, \lambda) = \int f_t(r, \rho)\lambda^{it} U_t \, dt$$

and the groupoid is parameterized by coordinates

$$\tilde{\mathcal{G}} = \{(r, \rho, \lambda) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} \times \mathbb{R}_+^* \mid r\rho \in \hat{\mathbb{Z}}\}.$$

The Bost–Connes algebra is $\mathcal{A} = C^*(\mathcal{G})$ with $\mathcal{G} = \tilde{\mathcal{G}}/\mathbb{R}_+^*$ the groupoid of the commensurability relation on 1-dimensional $\mathbb{Q}$-lattices up to scaling.

The combination of scaling and Galois action given an action of $\hat{\mathbb{Z}}^* \times \mathbb{R}_+^* = C_{\mathbb{Q}}$, since in the BC case the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ factors through the abelianization. Characters χ of $\hat{\mathbb{Z}}^*$ determine projectors $p_\chi = \int_{\hat{\mathbb{Z}}^*} g\chi(g)\, dg$ where p_χ is an idempotent in the category of endomotives and in $\mathrm{End}_\Lambda D(\mathcal{A}, \varphi)$. Thus, one can considered the cohomology

$$\mathrm{HC}_0(p_\chi \, D(\mathcal{A}, \varphi))$$

of the range of this projector acting on the cokernel $D(\mathcal{A}, \varphi)$ of the restriction map.

2.6 Scaling as Frobenius in characteristic zero. The observation that a scaling action appears to provide a natural replacement for the Frobenius in characteristic zero is certainly not new to the work of [11] described above. In fact, perhaps the first very strong evidence for the parallels between scaling and Frobenius came from the comparative analysis, given in §11 of [2] of the number theoretic, characteristic p method of Harder–Narasimhan [27] and the differential geometric method of Atiyah–Bott [2]. Both methods of [27] and [2] yield a computation of Betti numbers. In the number-theoretic setting this is achieved by counting points in the strata of a stratification, while in the Morse-theoretic approach one retracts strata onto the critical set. Both methods work because, on one side, one has a perfect Morse stratification, which essentially depends upon the fact that the strata are built out of affine spaces, and on the other hand one can effectively compute numbers of points in each stratum, for much the same reason. The explicit expressions obtained in both cases can be compared directly by a simple substitution that replaces the cardinality q by a real variable t^2 and the Frobenius eigenvalues ω_i by $-t^{-1}$. In following this parallel between the characteristic p and the characteristic zero case, one observes then that the role played by the Frobenius in the first setting is paralleled by a scaling action in the characteristic zero world.

More closely related to the specific setting of the BC endomotive, one knows from the result of [19] that there is an analog of the BC system for function fields, where one works exclusively in positive characteristic. This starts with the observation in [17] that the quantum statistical mechanical system of [14], generalizing the BC system to 2-dimensional $\mathbb{Q}$-lattices, can be equivalently formulated in terms of Tate modules of elliptic curves with marked points,

$$\mathrm{T}(E) = H^1(E, \hat{\mathbb{Z}}) \quad \text{with } \xi_1, \xi_2 \in \mathrm{T}(E),$$

with the commensurability relation implemented by isogenies. One then has a natural analog in the function field case. In fact, for $\mathbb{K} = \mathbb{F}_q(C)$, the usual equivalence of categories between elliptic curves and 2-dimensional lattices has an analog in terms of Drinfeld modules. One can then construct a noncommutative space, which can be described in terms of Tate modules of Drinfeld modules with marked points and the isogeny relation, or in the rank one case, in terms of 1-dimensional $\mathbb{K}$-lattices modulo commensurability. One constructs a convolution algebra, over a characteristic p field $\mathbb{C}_\infty$, which is the completion of the algebraic closure of the completion $\mathbb{K}_\infty$ at a point ∞ of C.

One can extend to positive characteristic some of the main notions of quantum statistical mechanics, by a suitable redefinition of the notion of time evolutions and of their analytic continuations, which enter in the definition of KMS states. Over complex numbers, for $\lambda \in \mathbb{R}_+^*$ and $s = x + iy$ one can exponentiate as $\lambda^s = \lambda^x e^{iy \log \lambda}$. In the function field context, there is a similar exponentiation, for positive elements (with respect to a sign function) in $\mathbb{K}_\infty^*$ and for $s = (x, y) \in S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$, with $\lambda^s = x^{\deg(\lambda)} \langle \lambda \rangle^y$, with $\deg(\lambda) = -d_\infty v_\infty(\lambda)$, where d_∞ is the degree of the point $\infty \in C$ and v_∞ the corresponding valuation, and $\lambda = \mathrm{sign}(\lambda) u_\infty^{v_\infty(\lambda)} \langle \lambda \rangle$ the decomposition analogous to the polar decomposition of complex numbers, involving a sign function and a uniformizer $\mathbb{K}_\infty = \mathbb{F}_{q^{d_\infty}}((u_\infty))$. The second term in the exponentiation is then given by

$$\langle \lambda \rangle^y = \sum_{j=0}^{\infty} \binom{y}{j} (\langle \lambda \rangle - 1)^j,$$

with the $\mathbb{Z}_p$-binomial coefficients

$$\binom{y}{j} = \frac{y(y-1)\dots(y-k+1)}{k!}.$$

Exponentiation is an entire function $s \mapsto \lambda^s$ from S_∞ to $\mathbb{C}_\infty^*$, with $\lambda^{s+t} = \lambda^s \lambda^t$, so one usually thinks of S_∞ as a function field analog of the complex line with its polar decomposition $\mathbb{C} = U(1) \times \mathbb{R}_+^*$. One can extend the above to exponentiate ideals, $I^s = x^{\deg(I)} \langle I \rangle^y$. This give an associated characteristic p valued zeta function, the Goss L-function of the function field,

$$Z(s) = \sum_I I^{-s},$$

which is convergent in a "half plane" of $\{s = (x, y) \in S_\infty \mid |x|_\infty > q\}$.

The analog of a time evolution in this characteristic p setting is then a continuous homomorphism $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$, where we think of $\mathbb{Z}_p$ as the line $\{s = (1, y) \in S_\infty\}$. In the case of the convolution algebra of 1-dimensional $\mathbb{K}$-lattices up to commensurability and scaling, a time evolution of this type is given using the exponentiation of ideals as

$$\sigma_y(f)(L, L') = \frac{\langle I \rangle^y}{\langle J \rangle^y} f(L, L'),$$

for pairs $L \sim L'$ of commensurable $\mathbb{K}$-lattices and the corresponding ideals. This gives a quantum statistical mechanical system in positive characteristic whose partition function is the Goss L-function.

One also has a notion of KMS_x functionals, which lack the positivity property of their characteristic zero version, but they have the defining property that $\varphi(ab) = \varphi(\sigma_x(b)a)$, where σ_x is the analytic continuation of the time evolution to $s = (x, 0)$.

Moreover, as shown in [19], one can construct a dual system in this function field setting as well, where the product on the dual algebra is constructed in terms of the momenta of the non-archimedean measure. The algebra of the dual system maps again naturally to the convolution algebra of the commensurability relation on 1-dimensional $\mathbb{K}$-lattices not up to scaling, which in turn can be expressed in terms of the adèles class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ of the function field. The algebra of the dual system also has a scaling action, exactly as in the characteristic zero case:

$$\theta_\lambda(X) = \int_H \ell(s)\lambda^s U_s d\mu(s),$$

where $H = G \times \mathbb{Z}_p$ with $G \subset \mathbb{C}_\infty^*$ and

$$\theta_\lambda|_G(X) = \theta_m(X) = \int \ell(s)x^{-d_\infty m} U_s 0, d\mu(s),$$

$$\theta_\lambda|_{\mathbb{Z}_p}(X) = \theta_{\langle\lambda\rangle} \int \ell(s)\langle\lambda\rangle^y U_s\, d\mu(s).$$

This action recovers the Frobenius action $Fr^{\mathbb{Z}}$ as the part $\theta_\lambda|_G$ of the scaling action, as well as the action of the inertia group, which corresponds to the part $\theta_\lambda|_{\mathbb{Z}_p}$.

2.7 The adèle class space and the Weil proof. The adèle class space is the bad quotient $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ of the adèles of a global field by the action of $\mathbb{K}^*$. Unlike the case of the action on the idèles $\mathbb{A}_{\mathbb{K}}^*$, which gives rise to a nice classical quotient, when one takes the action on the adèles the quotient is no longer described by a nice classical space, due to the ergodic nature of the action. However, it can be treated as a noncommutative space. In fact, this is the space underlying Connes' approach to the Riemann hypothesis via noncommutative geometry. Our purpose here is to describe the role of motivic ideas in noncommutative geometry, so we focus on the approach of [11] recalled in the previous section and we illustrate how the adèles class space relates to the algebra of the dual system of the Bost–Connes endomotive, as mentioned above for the function field analog.

A Morita equivalence given by compressing $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N} = (C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}_+^*)_\pi$ with the projection given by the characteristic function π of $\hat{\mathbb{Z}}$ can be used to identify the BC endomotive with the noncommutative quotient $\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}_+^*$. The dual system is then identified with the noncommutative quotient $\mathbb{A}_{\mathbb{Q}}^{\cdot}/\mathbb{Q}^*$, where $\mathbb{A}_{\mathbb{Q}}^{\cdot} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}^*$. The adèles class space $X_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ is obtained by adding the missing point $0 \in \mathbb{R}$.

The way in which the adèles class space entered in Connes' work [10] on the Riemann zeta function was through a sequence of Hilbert spaces

$$0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0 \xrightarrow{\mathfrak{E}} L^2_\delta(C_\mathbb{Q}) \to \mathcal{H} \to 0, \tag{2.1}$$

$$\mathfrak{E}(f)(g) = |g|^{1/2} \sum_{q\in\mathbb{Q}^*} f(qg) \quad \text{for all } g \in C_\mathbb{Q}, \tag{2.2}$$

where the space $L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0$ is defined by

$$0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*) \to \mathbb{C}^2 \to 0$$

imposing the conditions $f(0) = 0$ and $\hat{f}(0) = 0$. The sequence above is compatible with the $C_\mathbb{Q}$ actions, so the operators

$$U(h) = \int_{C_\mathbb{Q}} h(g) U_g \, d^*g, \quad h \in \mathcal{S}(C_\mathbb{Q}),$$

for compactly supported h, act on $\mathcal{H}$. The Hilbert space $\mathcal{H}$ can be decomposed $\mathcal{H} = \bigoplus_\chi \mathcal{H}_\chi$ according to characters χ of $\hat{\mathbb{Z}}^*$, and the scaling action of $\mathbb{R}^*_+$ on $\mathcal{H}_\chi = \{\xi \in \mathcal{H} \mid U_g\xi = \chi(g)\xi\}$ is generated by an operator D_χ with

$$\mathrm{Spec}(D_\chi) = \{s \in i\mathbb{R} \mid L_\chi(\tfrac{1}{2} + is) = 0\},$$

where L_χ is the L-function with Grössencharakter χ. In particular, the Riemann zeta function for $\chi = 1$.

The approach of Connes gives a semi-local trace formula, over the adèles class space restricted to a subset of finitely many places,

$$\mathrm{Tr}(R_\Lambda U(h)) = 2h(1) \log \Lambda + \sum_{v\in S} \int'_{\mathbb{Q}^*_v} \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$$

where R_Λ is a cutoff regularization and $\int'$ is the principal value.

The trace formula should be compared to the Weil's explicit formula in its distributional form:

$$\hat{h}(0) + \hat{h}(1) - \sum_\rho \hat{h}(\rho) = \sum_v \int'_{\mathbb{Q}^*_v} \frac{h(u^{-1})}{|1-u|} d^*u.$$

The geometric idea behind the Connes semi-local trace formula [10] is that it comes from the contributions of the periodic orbits of the action of $C_\mathbb{Q}$ on the complement of the classical points inside the adèles class space, $X_\mathbb{Q} \smallsetminus C_\mathbb{Q}$. These are counted according to a version of the Guillemin–Sternberg distributional trace formula, originally stated for a flow $F_t = \exp(tv)$ on manifold, implemented by transformations

$$(U_t f)(x) = f(F_t(x)), \quad f \in C^\infty(M).$$

Under a transversality hypothesis which gives $1 - (F_t)_*$ invertible, for

$$(F_t)_* \colon T_x/\mathbb{R}v_x \to T_x/\mathbb{R}v_x = N_x,$$

the distributional trace formula takes the form

$$\mathrm{Tr}_{\mathrm{distr}}\Big(\int h(t)U_t\,dt\Big) = \sum_\gamma \int_{I_\gamma} \frac{h(u)}{|1-(F_u)_*|}\,d^*u,$$

where γ ranges over periodic orbits and I_γ is the isotropy group, and d^*u a measure with $\mathrm{covol}(I_\gamma) = 1$. The distributional trace for a Schwartz kernel $(Tf)(x) = \int k(x,y)f(y)\,dy$ is of the form $\mathrm{Tr}_{\mathrm{distr}}(T) = \int k(x,x)\,dx$. For $(Tf)(x) = f(F(x))$, this gives $(Tf)(x) = \int \delta(y - F(x))f(y)\,dy$.

The work of [11] and [12] presents a different but closely related approach, where one reformulates the noncommutative geometry method of [10] in a *cohomological* form with a motivic flavor, as we explained in the previous sections.

The restriction morphism $\delta = (\mathrm{Tr}\circ\pi)^\natural$ from the dual system of the BC endomotive to its classical points, both seen as noncommutative motives in the category of cyclic modules, can be equivalently written as

$$\delta(f) = \sum_{n\in\mathbb{N}} f(1, n\rho, n\lambda) = \sum_{q\in\mathbb{Q}^*} \tilde{f}(q(\rho,\lambda)) = \mathfrak{E}(\tilde{f}),$$

where $\tilde{f}$ is an extension by zero outside of $\hat{\mathbb{Z}}\times\mathbb{R}^+ \subset \mathbb{A}_\mathbb{Q}$.

The Hilbert space $L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0$ is replaced here by the cyclic-module $\hat{\mathcal{A}}^\natural_{\beta,0}$. This requires different analytic techniques based on nuclear spaces, as in [47].

This provides a cohomological interpretation for the map $\mathfrak{E}$ and for the spectral realization, which is now associated to the scaling action θ on the cohomology $\mathrm{HC}_0(D(\mathcal{A},\varphi))$, which replaces the role of the Hilbert space $\mathcal{H}$ of (2.1).

One has an action of $C_\mathbb{Q} = \mathbb{A}^*_\mathbb{Q}/\mathbb{Q}^*$ on $\mathcal{H}^1 := \mathrm{HC}_0(D(\mathcal{A},\varphi))$ by

$$\vartheta(f) = \int_{C_\mathbb{Q}} f(g)\vartheta_g\,d^*g$$

for $f \in S(C_\mathbb{Q})$, a strong Schwartz space. The Weil's explicit formula then has a global trace formula interpretation as

$$\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1}) = \hat{f}(0) + \hat{f}(1) - \Delta\bullet\Delta f(1) - \sum_v \int'_{(\mathbb{K}_v^*, e_{K_v})} \frac{f(u^{-1})}{|1-u|} d^*u.$$

The term $\Delta\bullet\Delta = \log|a| = -\log|D|$, with D the discriminant for a number field, can be thought of as a self intersection of the diagonal, with the discriminant playing a role analogous to the Euler characteristic $\chi(C)$ of the curve for a function field $\mathbb{F}_q(C)$.

Thus, summarizing briefly the main differences between the approach of [10] and that of [11], [12], we have the following situation. In the trace formula for $\mathrm{Tr}(R_\Lambda U(f))$

of [10] only the zeros on critical line are involved and the Riemann hypothesis problem is equivalent to the problem of extending the semi-local trace formula to a global trace formula. This can be thought of in physical terms as a problem of passing from finitely many degrees of freedom to infinitely many, or equivalently from a quantum mechanical system to quantum field theory. In the setting of [11] and [12], instead, one has a global trace formula for $\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1})$ and all the zeros of the Riemann zeta function are involved, since one is no longer working in the Hilbert space setting that is biased in favor of the critical line. In this setting the Riemann hypothesis becomes equivalent to a *positivity* statement

$$\mathrm{Tr}\left(\vartheta(f \star f^{\sharp})|_{\mathcal{H}^1}\right) \geq 0 \quad \text{for all } f \in S(C_{\mathbb{Q}}),$$

where

$$(f_1 \star f_2)(g) = \int f_1(k) f_2(k^{-1}g) d^*g$$

with the multiplicative Haar measure d^*g and the adjoint is given by

$$f^{\sharp}(g) = |g|^{-1}\overline{f(g^{-1})}.$$

This second setting makes for a more direct comparison with the algebro-geometric and motivic setting of the Weil proof of the Riemann hypothesis for function fields, which is based on similar ingredients: the Weil explicit formula and a positivity statement for the trace of correspondences.

In a nutshell, the structure of the Weil proof for function fields is the following. The Riemann hypothesis for function fields $\mathbb{K} = \mathbb{F}_q(C)$ is the statement that the eigenvalues λ_n of the Frobenius have $|\lambda_j| = q^{1/2}$ in the zeta function

$$\zeta_{\mathbb{K}}(s) = \prod_{\Sigma_{\mathbb{K}}} (1 - q^{-n_v s})^{-1} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

with $P(T) = \prod(1-\lambda_n T)$ the characteristic polynomial of the Frobenius Fr^* acting on étale cohomology $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$. This statement is shown to be equivalent to a positivity statement $\mathrm{Tr}(Z \star Z') > 0$ for the trace of correspondences $Z = \sum_n a_n \mathrm{Fr}^n$ obtained from the Frobenius. Correspondences here are divisors $Z \subset C \times C$. These have a degree, codegree, and trace

$$d(Z) = Z \bullet (P \times C), \quad d'(Z) = Z \bullet (C \times P),$$
$$\mathrm{Tr}(Z) = d(Z) + d'(Z) - Z \bullet \Delta,$$

with Δ the diagonal in $C \times C$. One first adjusts the degree of the correspondence by trivial correspondences $C \times P$ and $P \times C$, then one applies Riemann–Roch to the divisor on the curve $P \mapsto Z(P)$ of deg $= g$ and shows that it is linearly equivalent to an effective divisor. Then using $d(Z \star Z') = d(Z)d'(Z) = gd'(Z) = d'(Z \star Z')$, one gets

$$\begin{aligned}\mathrm{Tr}(Z \star Z') &= 2gd'(Z) + (2g-2)d'(Z) - Y \bullet \Delta \\ &\geq (4g-2)d'(Z) - (4g-4)d'(Z) = 2d'(Z) \geq 0,\end{aligned}$$

where $Z \star Z' = d'(Z)\Delta + Y$.

In the noncommutative geometry setting of [11] and [12] the role of the Frobenius correspondences is played by the scaling action of elements $g \in C_{\mathbb{K}}$ by

$$Z_g = \{(x, g^{-1}x)\} \subset \mathbb{A}_{\mathbb{K}}/\mathbb{K}^* \times \mathbb{A}_{\mathbb{K}}/\mathbb{K}^* \tag{2.3}$$

and more generally $Z(f) = \int_{C_{\mathbb{K}}} f(g) Z_g \, d^*g$ with $f \in S(C_{\mathbb{K}})$. These correspondences also have a degree and codegree

$$d(Z(f)) = \hat{f}(1) = \int f(u)|u| \, d^*u$$

with $d(Z_g) = |g|$ and

$$d'(Z(f)) = d(Z(\bar{f}^{\sharp})) = \int f(u) \, d^*u = \hat{f}(0).$$

Adjusting degree $d(Z(f)) = \hat{f}(1)$ is possible by adding elements $h \in \mathcal{V}$, where $\mathcal{V}$ is the range of the restriction map $\delta = \mathrm{Tr} \circ \rho$,

$$h(u, \lambda) = \sum_{n \in \mathbb{Z}^\times} \eta(n\lambda)$$

with $\lambda \in \mathbb{R}_+^*$ and $u \in \hat{\mathbb{Z}}^*$, where $C_{\mathbb{Q}} = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^*$. Indeed, one can find an element $h \in \mathcal{V}$ with $\hat{h}(1) \neq 0$ since Fubini's theorem fails,

$$\int_{\mathbb{R}} \sum_n \eta(n\lambda) \, d\lambda \neq \sum_n \int_{\mathbb{R}} \eta(n\lambda) \, d\lambda = 0.$$

One does not have a good replacement in this setting for principal divisors and linear equivalence, although one expects that the role of Riemann–Roch should be played by an index theorem in noncommutative geometry.

3 Endomotives and $\mathbb{F}_1$-geometry

In trying to exploit the analogies between function fields and number fields to import some of the ideas and methods of the Weil proof to the number fields context, one of the main questions is whether one can construct a geometric object playing the role of the product $C \times_{\mathbb{F}_q} C$ over which the Weil argument with correspondences $Z = \sum a_n Fr^n$ is developed. We have seen in the previous section a candidate space built using noncommutative geometry, through the correspondences Z_g of (2.3) on the adèles class space. A different approach within algebraic geometry, aims at developing a geometry "over the field with one element" that would make it possible to interpret $\mathrm{Spec}(\mathbb{Z})$ as an analog of the curve, with a suitable space $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ playing the role of $C \times_{\mathbb{F}_q} C$.

The whole idea about a "field with one element", though no such thing can obviously exist in the usual sense, arises from early considerations of Tits on the behavior of the counting of points over finite fields in various examples of finite geometries. For instance, for $q = p^k$,

$$\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \smallsetminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q,$$
$$\#\,\mathrm{Gr}(n, j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\} = \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q,$$

where one sets

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q, \quad [0]_q! = 1.$$

In all this cases, the expression one obtains when setting $q = 1$ still makes sense and it appears to suggest a geometric replacement for each object. For example one obtains

$$\mathbb{P}^{n-1}(\mathbb{F}_1) := \text{ finite set of cardinality } n,$$
$$\mathrm{Gr}(n, j)(\mathbb{F}_1) := \text{ set of subsets of cardinality } j.$$

These observations suggest the existence of something like a notion of algebraic geometry over $\mathbb{F}_1$, even though one need not have a direct definition of $\mathbb{F}_1$ itself.

Further observations along these lines by Kapranov–Smirnov enriched the picture with a notion of "field extensions" $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$, which are described in terms of actions of the monoid $\{0\} \cup \mu_n$, with μ_n the group of n-th roots of unity.

In this sense, one can say that a vector space over $\mathbb{F}_{1^n}$ is a pointed set (V, v) endowed with a free action of μ_n on $V \smallsetminus \{v\}$ and linear maps are just permutations compatible with the action.

So, as observed by Soulé and Kapranov–Smirnov, although one does not define $\mathbb{F}_{1^n}$ and $\mathbb{F}_1$ directly, one can make sense of the change of coefficients from $\mathbb{F}_1$ to $\mathbb{Z}$ as

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^n - 1).$$

Various different approaches to $\mathbb{F}_1$-geometry have been developed recently by many authors: Soulé, Haran, Deitmar, Dourov, Manin, Toën–Vaquie, Connes–Consani, Borger, López-Peña and Lorscheid. Several of these viewpoints can be regarded as ways of providing *descent data* for rings from $\mathbb{Z}$ to $\mathbb{F}_1$. We do not enter here into a comparative discussion of these different approaches: a good overview of the current status of the subject is given in [39]. We are interested here in some of those versions of $\mathbb{F}_1$-geometry that can be directly connected with the noncommutative geometry approach described in the previous sections. We focus on the following approaches:

- descent data determined by cyclotomic points (Soulé [49]);
- descent data by Λ-ring structures (Borger [4]);
- analytic geometry over $\mathbb{F}_1$ (Manin [43]).

Soulé introduced in [49] a notion of *gadgets* over $\mathbb{F}_1$. These are triples of data $(X, \mathcal{A}_X, e_{x,\sigma})$, where $X : \mathcal{R} \to Sets$ is a covariant functor from a category $\mathcal{R}$ of finitely generated flat rings, which can be taken to be the subcategory of rings generated by the group rings $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$, $\mathcal{A}_X$ is a complex algebra with evaluation maps $e_{x,\sigma}$) such that, for all $x \in X(R)$ and $\sigma : R \to \mathbb{C}$, one has an algebra homomorphism $e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$ with

$$e_{f(y),\sigma} = e_{y,\sigma\circ f}$$

for any ring homomorphism $f : R' \to R$.

For example, affine varieties $V_{\mathbb{Z}}$ over $\mathbb{Z}$ define gadgets $X = G(V_{\mathbb{Z}})$, by setting $X(R) = \mathrm{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$.

An affine variety over $\mathbb{F}_1$ is then a gadget with $X(R)$ finite, and a variety $X_{\mathbb{Z}}$ with a morphism of gadgets $X \to G(X_{\mathbb{Z}})$, with the property that, for all morphisms $X \to G(V_{\mathbb{Z}})$, there exists a unique algebraic morphism $X_{\mathbb{Z}} \to V_{\mathbb{Z}}$ that functorially corresponds to the morphism of gadgets.

The Soulé data can be thought of as a descent condition from $\mathbb{Z}$ to $\mathbb{F}_1$, by regarding them as selecting among varieties defined over $\mathbb{Z}$ those that are determined by the data of their cyclotomic points $X(R)$, for $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$. This selects varieties that are very combinatorial in nature. For example, smooth toric varieties are geometries over $\mathbb{F}_1$ in this and all the other currently available flavors of $\mathbb{F}_1$-geometry.

Borger's approach to $\mathbb{F}_1$-geometry in [4] is based on a different way of defining descent conditions from $\mathbb{Z}$ to $\mathbb{F}_1$, using lifts of Frobenia, encoded in the algebraic structure of Λ-rings. This was developed by Grothendieck in the context of characteristic classes and the Riemann–Roch theorem, where it relates to operations in K-theory, but it can be defined abstractly in the following way.

For a ring R, whose underlying abelian group is torsion free, a Λ-ring structure is an action of the multiplicative semigroup $\mathbb{N}$ of positive integers by endomorphisms lifting Frobenius, namely, such that

$$s_p(x) - x^p \in pR \quad \text{for all } x \in R.$$

Morphisms of Λ-rings are ring homomorphisms $f : R \to R'$ compatible with the actions, $f \circ s_k = s'_k \circ f$.

The Bost–Connes endomotive, which is at the basis of the noncommutative geometry approach to the Riemann hypothesis, relates directly to both of these notions of $\mathbb{F}_1$ geometry in a very natural way.

3.1 Endomotives and Soulé's $\mathbb{F}_1$-geometry. The relation between the BC endomotive and Soulé's $\mathbb{F}_1$ geometry was investigated in [13]. One first considers a model over $\mathbb{Z}$ of the BC algebra. This requires eliminating denominators from the relations of the algebra over $\mathbb{Q}$. It can be done by replacing the crossed product by ring endomorphisms by a more subtle "crossed product" by correspondences. More precisely, one considers the algebra $\mathcal{A}_{\mathbb{Z},\mathrm{BC}}$ generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and elements μ_n^*, $\tilde{\mu}_n$ with relations

$$\begin{aligned}\tilde{\mu}_n\tilde{\mu}_m &= \tilde{\mu}_{nm},\\ \mu_n^*\mu_m^* &= \mu_{nm}^*,\\ \mu_n^*\tilde{\mu}_n &= n,\\ \tilde{\mu}_n\mu_m^* &= \mu_m^*\tilde{\mu}_n, \quad (n,m)=1.\end{aligned}$$

One has

$$\mu_n^* x = \sigma_n(x)\mu_n^* \quad \text{and} \quad x\tilde{\mu}_n = \tilde{\mu}_n\sigma_n(x),$$

where $\sigma_n(e(r)) = e(nr)$ for $r \in \mathbb{Q}/\mathbb{Z}$.

Notice that here the ring homomorphisms $\rho_n(x) = \mu_n x \mu_n^*$ are replaced by $\tilde{\rho}_n(x) = \tilde{\mu}_n x \mu_n^*$, which are no longer ring homomorphisms, but correspondences. The resulting "crossed product" is indicated by the notation $\mathcal{A}_{\mathbb{Z},\mathrm{BC}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$.

One then observes that roots of unity

$$\underline{\mu}^{(k)}(R) = \{x \in R \mid x^k = 1\} = \mathrm{Hom}_{\mathbb{Z}}(A_k, R)$$

with $A_k = \mathbb{Z}[t,t^{-1}]/(t^k-1)$ can be organized as a system of varieties over $\mathbb{F}_1$ in two different ways. As an inductive system they define the multiplicative group $\mathbb{G}_m$ as a variety over $\mathbb{F}_1$ by taking

$$\underline{\mu}^{(n)}(R) \subset \underline{\mu}^{(m)}(R), \quad n|m, \quad A_m \twoheadrightarrow A_n,$$

and by taking the complex algebra to be $\mathcal{A}_X = C(S^1)$.

As a projective system, which corresponds to the BC endomotive, one uses the morphisms $\xi_{m,n}\colon X_n \twoheadrightarrow X_m$,

$$\xi_{m,n}\colon \underline{\mu}^{(n)}(R) \twoheadrightarrow \underline{\mu}^{(m)}(R), \quad n|m,$$

and obtains a pro-variety

$$\underline{\mu}^{\infty}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R),$$

which arises from the projective system of affine varieties over $\mathbb{F}_1$,

$$\xi_{m,n}\colon \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} \to \mathbb{F}_{1^m} \otimes_{\mathbb{F}_1} \mathbb{Z},$$

where the complex algebra is taken to be $\mathcal{A}_X = \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$.

The affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ are defined by gadgets $G(\mathrm{Spec}(\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]))$, which form a projective system of gadgets. The endomorphisms σ_n of varieties over $\mathbb{Z}$, are also endomorphisms of gadgets and of $\mathbb{F}_1$-varieties.

The extensions $\mathbb{F}_{1^n}$ of Kapranov–Smirnov correspond to the free actions of roots of unity

$$\zeta \mapsto \zeta^n,\ n \in \mathbb{N}, \quad \text{and} \quad \zeta \mapsto \zeta^\alpha \leftrightarrow e(\alpha(r)),\ \alpha \in \hat{\mathbb{Z}},$$

and these can be regarded as the Frobenius action on $\mathbb{F}_{1^\infty}$. It should be noted that, indeed, in reductions mod p of the integral Bost–Connes endomotive these do correspond to the Frobenius, that one can consider the BC endomotive as describing the tower of extensions $\mathbb{F}_{1^n}$ together with the Frobenius action.

As shown in [13], one can obtain characteristic p versions of the BC endomotive by separating out the parts

$$\mathbb{Q}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$$

with denominators that are powers of p and denominators that are prime to p. One then has a crossed product algebra

$$\mathbb{K}[\mathbb{Q}_p/\mathbb{Z}_p] \rtimes p^{\mathbb{Z}^+}$$

with endomorphisms σ_n for $n = p^\ell$ and $\ell \in \mathbb{Z}^+$.

The Frobenius $\varphi_{\mathbb{F}_p}(x) = x^p$ of the field $\mathbb{K}$ in characteristic p satisfies

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(f) = f^{p^\ell}$$

for $f \in \mathbb{K}[\mathbb{Q}/\mathbb{Z}]$ so that one has

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(e(r) \otimes x) = e(p^\ell r) \otimes x^{p^\ell} = (e(r) \otimes x)^{p^\ell}.$$

This shows that the BC endomorphisms restrict to Frobenia on the mod p reductions of the system: σ_{p^ℓ} induces the Frobenius correspondence on the pro-variety $\mu^\infty \otimes_{\mathbb{Z}} \mathbb{K}$.

3.2 Endomotives and Borger's $\mathbb{F}_1$-geometry via Λ-rings. This is also the key observation in relating the BC endomotive to Borger's point of view [4] on $\mathbb{F}_1$-geometry. One sees, in fact, that the Bost–Connes endomotive is a direct limit of Λ-rings

$$R_n = \mathbb{Z}[t, t^{-1}]/(t^n - 1), \quad s_k(P)(t, t^{-1}) = P(t^k, t^{-k}),$$

where the Λ-ring structures is given by the endomorphisms action of $\mathbb{N}$. The action of $\hat{\mathbb{Z}}$ that combines the action of symmetries $\hat{\mathbb{Z}}^*$ by automorphisms of the BC system and the endomorphisms that give the Λ-ring structure is given by

$$\alpha \in \hat{\mathbb{Z}} \colon (\zeta \colon x \mapsto \zeta x) \mapsto (\zeta \colon x \mapsto \zeta^\alpha x),$$

where we identify $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. This agrees with the notion of Frobenius over $\mathbb{F}_{1^\infty}$ proposed by Haran.

In fact, one can see the relation to Λ-rings more precisely by introducing multivariable generalizations of the BC endomotive as in [45].

One considers as varieties the algebraic tori $\mathbb{T}^n = (\mathbb{G}_m)^n$ with endomorphisms $\alpha \in M_n(\mathbb{Z})^+$ and one constructs, as in the case of the BC endomotive, the preimages

$$X_\alpha = \{t = (t_1, \dots, t_n) \in \mathbb{T}^n \mid s_\alpha(t) = t_0\}$$

organized into a projective system with maps

$$\xi_{\alpha,\beta} : X_\beta \to X_\alpha, \quad t \mapsto t^\gamma, \quad \alpha = \beta\gamma \in M_n(\mathbb{Z})^+,$$
$$t \mapsto t^\gamma = \sigma_\gamma(t) = (t_1^{\gamma_{11}} t_2^{\gamma_{12}} \dots t_n^{\gamma_{1n}}, \dots, t_1^{\gamma_{n1}} t_2^{\gamma_{n2}} \dots t_n^{\gamma_{nn}}).$$

The projective limit $X = \varprojlim_\alpha X_\alpha$ carries a semigroup action of $M_n(\mathbb{Z})^+$.

One can then consider the algebra $C(X(\bar{\mathbb{Q}})) \cong \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ with generators $e(r_1) \otimes \dots \otimes e(r_n)$ and the crossed product

$$\mathcal{A}_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and μ_α, μ_α^* with

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r}) \mu_\alpha^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s}) = \underline{r}} e(\underline{s}),$$
$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha = e(\alpha(\underline{r})).$$

This corresponds to the action of the family of endomorphisms

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha.$$

These multivariable versions relate to the Λ-rings notion of $\mathbb{F}_1$-geometry through a theorem of Borger–de Smit, which shows that every torsion free finite rank Λ-ring embeds in a finite product of copies of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, where the action of $\mathbb{N}$ is compatible with the diagonal action $S_{n,\mathrm{diag}} \subset M_n(\mathbb{Z})^+$ in the multivariable BC endomotives. Thus, the multivariable BC endomotives are universal for Λ-rings.

3.3 Endomotives and Manin's analytic geometry over $\mathbb{F}_1$. These multivariable generalizations of the BC endomotive introduced in [45] are also closely related to Manin's approach to analytic geometry over $\mathbb{F}_1$ of [43], which is based on the Habiro ring as a ring of analytic functions of roots of unity. The Habiro ring [26] is defined as the projective limit

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n),$$

where

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n)$$

and one has morphisms $\mathbb{Z}[q]/((q)_n) \twoheadrightarrow \mathbb{Z}[q]/((q)_k)$ for $k \le n$ since $(q)_k | (q)_n$. This ring has evaluation maps at roots of unity that are surjective ring homomorphisms

$$\mathrm{ev}_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta],$$

but which, combined, give an *injective* homomorphism

$$\mathrm{ev}\colon \widehat{\mathbb{Z}[q]} \to \prod_{\zeta\in\mathcal{Z}} \mathbb{Z}[\zeta].$$

The elements of the Habiro ring also have Taylor series expansions at all roots of unity

$$\tau_\zeta\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]],$$

which also are *injective* ring homomorphisms. Thus, they behave like "analytic functions on roots of unity".

As argued in [45], the Habiro ring provides then another model for the noncommutative geometry of the cyclotomic tower, replacing $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ with $\widehat{\mathbb{Z}[q]}$.

One considers endomorphisms $\sigma_n(f)(q) = f(q^n)$, which lift $P(\zeta) \mapsto P(\zeta^n)$ in $\mathbb{Z}[\zeta]$ through the evaluation maps ev_ζ. This gives an action of $\mathbb{N}$ by endomorphisms and one can form a group crossed product

$$\mathcal{A}_{\mathbb{Z},q} = \widehat{\mathbb{Z}[q]}_\infty \rtimes \mathbb{Q}^*_+,$$

where $\mathcal{A}_{\mathbb{Z},q}$ is generated by $\widehat{\mathbb{Z}[q]}$ and by elements μ_n and μ_n^* with

$$\mu_n\sigma_n(f) = f\mu_n, \quad \mu_n^* f = \sigma_n(f)\mu_n^*.$$

The ring $\widehat{\mathbb{Z}[q]}_\infty = \bigcup_N A_N$, with A_N generated by the $\mu_N f \mu_N^*$, satisfies

$$\widehat{\mathbb{Z}[q]}_\infty = \varinjlim_n (\sigma_n\colon \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}).$$

These maps are injective and determine *automorphisms* $\sigma_n\colon \widehat{\mathbb{Z}[q]}_\infty \to \widehat{\mathbb{Z}[q]}_\infty$. Another way to describe this is in terms of the ring $\mathcal{P}_\mathbb{Z}$ of polynomials in $\mathbb{Q}$-powers q^r. One has

$$\hat{\mathcal{P}}_\mathbb{Z} = \varprojlim_N \mathcal{P}_\mathbb{Z}/\mathcal{J}_N,$$

where $\mathcal{J}_N$ is the ideal generated by $(q^r)_N = (1-q^r)\dots(1-q^{rN})$, with $r \in \mathbb{Q}^*_+$, and

$$\widehat{\mathbb{Z}[q]}_\infty \simeq \hat{\mathcal{P}}_\mathbb{Z}, \quad \mu_n f \mu_n^* \mapsto f(q^{1/n}),$$

where $\rho_r(f)(q) = f(q^r)$.

In [43], Manin also introduced multivariable versions of the Habiro ring,

$$\widehat{\mathbb{Z}[q_1,\dots,q_n]} = \varprojlim_N \mathbb{Z}[q_1,\dots,q_n]/I_{n,N}, \tag{3.1}$$

where $I_{n,N}$ is the ideal

$$((q_1-1)(q_1^2-1)\dots(q_1^N-1),\dots,(q_n-1)(q_n^2-1)\dots(q_n^N-1)).$$

These again have evaluations at roots of unity

$$\mathrm{ev}_{(\zeta_1,\dots,\zeta_n)}\colon \widehat{\mathbb{Z}[q_1,\dots,q_n]} \to \mathbb{Z}[\zeta_1,\dots,\zeta_n]$$

and Taylor series expansions

$$T_Z\colon \widehat{\mathbb{Z}[q_1,\dots,q_n]} \to \mathbb{Z}[\zeta_1,\dots,\zeta_n][[q_1-\zeta_1,\dots,q_n-\zeta_n]]$$

for all $Z = (\zeta_1,\dots,\zeta_n)$ in $\mathcal{Z}^n$ with $\mathcal{Z}$ the set of all roots of unity.

One can equivalently describe (3.1) as

$$\widehat{\mathbb{Z}[q_1,\dots,q_n]} = \varprojlim_N \mathbb{Z}[q_1,\dots,q_n,q_1^{-1},\dots,q_n^{-1}]/\mathcal{J}_{n,N},$$

where $\mathcal{J}_{n,N}$ is the ideal generated by the $(q_i-1)\dots(q_i^N-1)$, for $i=1,\dots,n$ and the $(q_i^{-1}-1)\dots(q_i^{-N}-1)$. Consider then again the algebraic tori $\mathbb{T}^n = (\mathbb{G}_m)^n$, with algebra $\mathbb{Q}[t_i, t_i^{-1}]$. Using the notation

$$t^\alpha = (t_i^\alpha)_{i=1,\dots,n} \quad \text{with } t_i^\alpha = \textstyle\prod_j t_j^{\alpha_{ij}},$$

we can define the semigroup action of $\alpha \in M_n(\mathbb{Z})^+$ by

$$q \mapsto \sigma_\alpha(q) = \sigma_\alpha(q_1,\dots,q_n) = (q_1^{\alpha_{11}} q_2^{\alpha_{12}} \dots q_n^{\alpha_{1n}}, \dots, q_1^{\alpha_{n1}} q_2^{\alpha_{n2}} \dots q_n^{\alpha_{nn}}) = q^\alpha,$$

analogous to the case of the multivariable BC endomotives discussed above, of which these constitute an analog in the setting of analytic $\mathbb{F}_1$-geometry.

4 DG-algebras and noncommutative motives

The noncommutative motives we encountered so far in this overview are derived from two sources: the abelian category of cyclic modules and the category of endomotives. The latter are a very special kind of zero-dimensional noncommutative spaces combining Artin motives and endomorphism actions. More generally, one would like to incorporate higher dimensional algebraic varieties and correspondences given by algebraic cycles, together with their self-maps, and construct larger categories of noncommutative spaces that generalize whet we saw here in a zero-dimensional setting. In particular, this would be needed in order to generalize some of the results obtained so far for the Riemann zeta function using noncommutative geometry, like the trace formulae discussed above, to the more general context of L-functions of algebraic varieties and motives.

When one wishes to combine higher dimensional algebraic varieties with noncommutative spaces, one needs to pay attention to the substantially different way in which one treats the rings of functions in the two settings. This is not visible in a purely zero-dimensional case where one deals only with Artin motives. When one treats noncommutative spaces as algebras, one point of view is that one essentially only needs to deal with the affine case. The reason behind this is the fact that the way to describe in noncommutative geometry the gluing of affine charts, or any other kind of identification, is by considering the convolution algebra of the equivalence relation that implements the identifications. So, at the expense of no longer working with commutative algebras, one gains the possibility of always working with a single algebra of functions.

When one tries to combine noncommutative spaces with algebraic varieties, however, one wants to be able to deal directly with the algebro-geometric description of arbitrary quasi-projective varieties. This is where a more convenient approach is provided by switching the point of view from algebras to categories. The main result underlying the categorical approach to combining noncommutative geometry and motives is the fact that the derived category $\mathcal{D}(X)$ of quasicoherent sheaved on a quasiseparated quasicompact scheme X is equivalent to the derived category $\mathcal{D}(\mathcal{A}^\bullet)$ of a DG-algebra $\mathcal{A}^\bullet$, which is unique up to derived Morita equivalence, see [3], [35]. Thus, passing to the setting of DG-algebras and DG-category provides a good setting where algebraic varieties can be treated, up to derived Morita equivalence, as noncommutative spaces.

A related question is the notion of correspondences between noncommutative spaces. We have seen in this short survey different notions of correspondences: morphisms of cyclic modules, among which one finds morphisms of algebras, bimodules, Morita equivalences, and traces. We also saw the correspondences associated to the scaling action of $C_{\mathbb{K}}$ on the noncommutative adèles class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$. More generally, the problem of identifying the best class of morphisms of noncommutative spaces (or better of noncommutative motives) that accounts for all the desired features remains a question that is not settled in a completely satisfactory ways. A comparison between different notions of correspondences in the analytic setting of KK-theory and in the context of derived algebraic geometry was given recently in [40], while a "motivic" category with correspondences based on noncommutative spaces defined as spectral triples and a version of smooth KK-theory was proposed in [46]. Again, a closer interplay between the analytic approach to noncommutative geometry via algebras, KK-theory, spectral triples, and such smooth differential notions, and the algebro-geometric approach via DG-categories and derived algebraic geometry is likely to play a crucial role in identifying the best notion of correspondences in noncommutative geometry.

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A short survey on pre-Lie algebras

Dominique Manchon

Contents

Introduction

A *left pre-Lie algebra* over a field k is a k-vector space A with a bilinear binary composition $\rhd$ that satisfies the left pre-Lie identity

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c),$$

for $a, b, c \in A$. Analogously, a *right pre-Lie algebra* is a k-vector space A with a binary composition $\lhd$ that satisfies the right pre-Lie identity

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b).$$

The left pre-Lie identity rewrites as

$$L_{[a,b]} = [L_a, L_b], \tag{1}$$

where $L_a \colon A \to A$ is defined by $L_a b = a \rhd b$, and where the bracket on the left-hand side is defined by $[a,b] := a \rhd b - b \rhd a$. As a consequence this bracket satisfies the Jacobi identity.

Pre-Lie algebras are sometimes called *Vinberg algebras*, as they appear in the work of E. B. Vinberg [33] under the name “left-symmetric algebras” on the classification of homogeneous cones. They appear independently at the same time in the work of M. Gerstenhaber [22] on Hochschild cohomology and deformations of algebras, under the name “pre-Lie algebras” which is now the standard terminology. Note however that Gerstenhaber’s pre-Lie algebras live in the category of graded vector spaces, and then additional signs occur in the left pre-Lie identity. The term “chronological algebras” has also been sometimes used, e.g., in the fundamental work of A. Agrachev and R. Gamkrelidze [1].

As we shall see, the notion itself can be however traced back to the work of A. Cayley [6] which, in modern language, describes *the* pre-Lie algebra morphism F_a from the pre-Lie algebra of rooted trees into the pre-Lie algebra of vector fields on $\mathbb{R}^n$ sending the one-vertex tree to a given vector field a. A very important step was indeed made at the turn of the century, with the explicit description of free pre-Lie algebras as pre-Lie algebras of decorated rooted trees endowed with grafting. F. Chapoton and M. Livernet [10] obtained this result as a byproduct of their complete description of the operad governing pre-Lie algebras in terms of labelled rooted trees. A. Dzhumadil'daev and C. Löfwall [14] give a more "elementary" proof, avoiding the language of operads, and providing in the same time the explicit description, again in terms of rooted trees, of the free NAP algebra.[1]

An important family of examples of pre-Lie algebras is given by any augmented operad, summing up the partial compositions and quotienting by the actions of the symmetric groups [8], [11]. Another source of examples is given by Rota–Baxter algebras, and more generally by Loday's dendriform algebras and twisted versions of those [15], [19], [17].

The article is organized as follows: in the first part we recall the main definitions and properties, following [1]. In particular we introduce the group of formal flows of a complete filtered pre-Lie algebra and we prove a pre-Lie version of the Poincaré–Birkhoff–Witt theorem. We also recall a recent theorem by J-L. Loday and M. Ronco relating left pre-Lie algebras and right-sided commutative Hopf algebras.

The second part is devoted to rooted trees and the free pre-Lie algebra. Following [10] and [8] we recall the construction of the group associated to any augmented operad as well as the associated pre-Lie algebra, and we recall the description of the free pre-Lie algebra in terms of rooted trees. As an application we describe a second pre-Lie algebra product on rooted trees which acts on the first by derivations [5], [30].

The third part is devoted to pre-Lie algebras of vector fields on $\mathbb{R}^n$, from the work of A. Cayley [6] to modern developments in numerical analysis through B-series [4], [24], [31], [12]. Finally the last part reviews the relation of pre-Lie algebras with other algebraic structures.

The reader will find a complementary point of view on pre-Lie algebras in the survey article of D. Burde [3], where emphasis is made on geometric aspects through affine structures on Lie groups.

1 Pre-Lie algebras

As any right pre-Lie algebra $(A, \lhd)$ is also a left pre-Lie algebra with product $a \rhd b := b \lhd a$, one can stick to left pre-Lie algebras, which we shall do unless specifically indicated.

[1]NAP stands for Non-Associative Permutative. A (left) NAP algebra is a vector space endowed with a bilinear map $\circ$ such that $a \circ (b \circ c) = b \circ (a \circ c)$. Right NAP algebras are defined accordingly. Left (right) NAP algebras are called "left (right) commutative" in [14].

1.1 The group of formal flows. The following is taken from the paper of A. Agrachev and R. Gamkrelidze [1]. Suppose that A is a left pre-Lie algebra endowed with a compatible decreasing filtration, namely $A = A_1 \supset A_2 \subset A_3 \supset \cdots$, such that the intersection of the A_j's reduces to $\{0\}$, and such that $A_p \rhd A_q \subset A_{p+q}$. Suppose moreover that A is complete with respect to this filtration. The Baker–Campbell–Hausdorff formula

$$C(a,b) = a + b + \frac{1}{2}[a,b] + \frac{1}{12}([a,[a,b]] + [b,[b,a]]) + \cdots$$

then endows A with a structure of pro-unipotent group. This group admits a more transparent presentation as follows: introduce a fictitious unit $\mathbf{1}$ such that $\mathbf{1} \rhd a = a \rhd \mathbf{1} = a$ for any $a \in A$, and define $W : A \to A$ by

$$W(a) := e^{L_a}\mathbf{1} - \mathbf{1} = a + \frac{1}{2}a \rhd a + \frac{1}{6}a \rhd (a \rhd a) + \cdots .$$

The mapping W is clearly a bijection. The inverse, denoted by Ω, also appears under the name "pre-Lie Magnus expansion" in [18]. It satisfies the equation

$$\Omega(a) = \frac{L_{\Omega(a)}}{e^{L_{\Omega(a)}} - Id}a = \sum_{i \geq 0} B_i L^i_{\Omega(a)} a,$$

where the B_i's are the Bernoulli numbers. The first few terms are

$$\Omega(a) = a - \frac{1}{2}a \rhd a + \frac{1}{4}(a \rhd a) \rhd a + \frac{1}{12}a \rhd (a \rhd a) + \cdots .$$

Transferring the BCH product by means of the map W, namely

$$a \,\#\, b = W(C(\Omega(a), \Omega(b))), \tag{2}$$

we have $W(a) \,\#\, W(b) = W\big(C(a,b)\big) = e^{L_a}e^{L_b}\mathbf{1} - \mathbf{1}$, hence $W(a) \,\#\, W(b) = W(a) + e^{L_a}W(b)$. The product # is thus given by the simple formula

$$a \,\#\, b = a + e^{L_{\Omega(a)}}b.$$

The inverse is given by $a^{\#-1} = W\big(-\Omega(a)\big) = e^{-L_{\Omega(a)}}\mathbf{1} - \mathbf{1}$. If $(A, \rhd)$ and $(B, \rhd)$ are two such pre-Lie algebras and $\psi : A \to B$ is a filtration-preserving pre-Lie algebra morphism, it is immediate to check that for any $a, b \in A$ we have

$$\psi(a \,\#\, b) = \psi(a) \,\#\, \psi(b).$$

In other words, the group of formal flows is a functor from the category of complete filtered pre-Lie algebras to the category of groups.

When the pre-Lie product $\rhd$ is associative, all this simplifies to

$$a \,\#\, b = a \rhd b + a + b$$

and

$$a^{\#-1} = \frac{1}{1+a} - 1 = \sum_{n \geq 1} (-1)^n a_n.$$

1.2 The pre-Lie Poincaré–Birkhoff–Witt theorem

Theorem 1.1. *Let A be any left pre-Lie algebra, and let $S(A)$ be its symmetric algebra, i.e., the free commutative algebra on A. Let A_{Lie} be the underlying Lie algebra of A, i.e., the vector space A endowed with the Lie bracket given by $[a,b] = a \rhd b - b \rhd a$ for any $a, b \in A$, and let $\mathcal{U}(A)$ be the enveloping algebra of A_{Lie}, endowed with its usual increasing filtration. Let us consider the associative algebra $\mathcal{U}(A)$ as a left module over itself.*

There exists a left $\mathcal{U}(A)$-module structure on $S(A)$ and a canonical left $\mathcal{U}(A)$-module isomorphism $\eta\colon \mathcal{U}(A) \to \mathcal{S}(A)$ such that the associated graded linear map $\mathrm{Gr}\,\eta\colon \mathrm{Gr}\,\mathcal{U}(A) \to S(A)$ is an isomorphism of commutative graded algebras.

Proof. The Lie algebra morphism

$$L\colon A \to \mathrm{End}\,A, \quad a \mapsto (L_a\colon b \mapsto a \rhd b),$$

extends by the Leibniz rule to a unique Lie algebra morphism $L\colon A \to \mathrm{Der}\,S(A)$. Now we claim that the map $M\colon A \to \mathrm{End}\,S(A)$ defined by

$$M_a u = au + L_a u$$

is a Lie algebra morphism. Indeed, we have for any $a, b \in A$ and $u \in S(A)$:

$$\begin{aligned} M_a M_b u &= M_a(bu + L_b u) \\ &= abu + aL_b u + L_a(bu) + L_a L_b u \\ &= abu + aL_b u + bL_a u + (a \rhd b)u + L_a L_b u. \end{aligned}$$

Hence

$$[M_a, M_b]u = (a \rhd b - b \rhd a)u + [L_a, L_b]u = M_{[a,b]}u,$$

which proves the claim. Now M extends, by the universal property of the enveloping algebra, to a unique algebra morphism $M\colon \mathcal{U}(A) \to \mathrm{End}\,S(A)$. The linear map

$$\eta\colon \mathcal{U}(A) \to S(A), \quad u \mapsto M_u \cdot 1,$$

is clearly a morphism of left $\mathcal{U}(A)$-modules. It is immediately seen by induction that for any $a_1, \ldots, a_n \in A$ we have $\eta(a_1 \ldots a_n) = a_1 \ldots a_n + v$ where v is a sum of terms of degree $\leq n-1$. This proves the theorem. □

Remark 1.2. Let us recall that the symmetrization map $\sigma\colon \mathcal{U}(A) \to S(A)$, uniquely determined by $\sigma(a^n) = a^n$ for any $a \in A$ and any integer n, is an isomorphism for the two A_{Lie}-module structures given by the adjoint action. This is *not* the case for the map η defined above. The fact that it is possible to replace the adjoint action of $\mathcal{U}(A)$ on itself by the simple left multiplication is a remarkable property of pre-Lie algebras, and makes Theorem 1.1 different from the usual Lie algebra PBW theorem.

Let us finally notice that if p stands for the projection from $S(A)$ onto A, for any $a_1, \dots, a_k \in A$ we easily get

$$p \circ \eta(a_1 \dots a_k) = L_{a_1} \dots L_{a_k} \mathbf{1} = a_1 \rhd (a_2 \rhd (\dots (a_{k-1} \rhd a_k) \dots)) \tag{3}$$

by a simple induction on k. The linear isomorphism η transfers the product of the enveloping algebra $\mathcal{U}(A)$ into a noncommutative product $*$ on $\mathcal{S}(A)$ defined by

$$s * t = \eta(\eta^{-1}(s)\eta^{-1}(t)).$$

Suppose now that A is endowed with a complete decreasing compatible filtration as in Section 1.1. This filtration induces a complete decreasing filtration $S(A) = S(A)_0 \supset S(A)_1 \supset S(A)_2 \supset \cdots$, and the product $*$ readily extends to the completion $\widehat{S(A)}$. For any $a \in A$, the application of (3) gives

$$p(e^{*a}) = W(a)$$

as an equality in the completed symmetric algebra $\widehat{S(A)}$.

According to (2) we can identify the pro-unipotent group $\{e^{*a},\ a \in A\} \subset \widehat{S(A)}$ and the group of formal flows of the pre-Lie algebra A by means of the projection p, namely,

$$p(e^{*a}) \# p(e^{*b}) = p(e^{*a} * e^{*b})$$

for any $a, b \in A$.

1.3 Right-sided commutative Hopf algebras and the Loday–Ronco theorem. Let $\mathcal{H}$ be a commutative Hopf algebra. Following [27], we say that $\mathcal{H}$ is *right-sided* if it is free as a commutative algebra, i.e., $\mathcal{H} = S(V)$ for some k-vector space V, and if the coproduct satisfies

$$\Delta(V) \subset \mathcal{H} \otimes V.$$

Suppose moreover that $V = \bigoplus_{n \geq 1} V_n$ is graded with finite-dimensional homogeneous components. Then the graded dual $A = V^0$ is a left pre-Lie algebra, and by the Milnor–Moore theorem, the graded dual $\mathcal{H}^0$ is isomorphic to the enveloping algebra $\mathcal{U}(A_{\mathrm{Lie}})$ as graded Hopf algebra. Conversely, for any graded pre-Lie algebra A the graded dual $\mathcal{U}(A_{\mathrm{Lie}})^0$ is free commutative right-sided ([27] Theorem 5.3).

2 Operads, pre-Lie algebras and rooted trees

We first associate a (right) pre-Lie algebra to any augmented operad, following F. Chapoton [8], and then we recall the description of the pre-Lie operad itself by F. Chapoton and M. Livernet [10], thus leading to two pre-Lie structures on the vector space generated by the rooted trees.

2.1 Pre-Lie algebras associated to augmented operads. Recall that an augmented operad $\mathcal{P}$ (in the symmetric monoidal category of vector spaces over some field k) is given by a collection of vector spaces $(\mathcal{P}_n)_{n\geq 1}$ with $\mathcal{P}_1 = k \cdot e$, an action of the symmetric group S_n on $\mathcal{P}_n$, and a collection of *partial compositions*

$$\circ_i : \mathcal{P}_k \otimes \mathcal{P}_l \to \mathcal{P}_{k+l-1}, \quad (a,b) \longmapsto a \circ_i b, \quad i = 1, \ldots, k,$$

which, for any $a \in \mathcal{P}_k, b \in \mathcal{P}_l, a \in \mathcal{P}_m$ satisfies the associativity conditions

$$(a \circ_i b) \circ_{i+j-1} c = a \circ_i (b \circ_j c), \quad i \in \{1, \ldots, k\},\ j \in \{1, \ldots, l\},$$
$$(a \circ_i b) \circ_{l+j-1} c = (a \circ_j c) \circ_i b, \quad i, j \in \{1, \ldots, k\},\ i < j,$$

the unit axiom

$$e \circ a = a, \quad a \circ_i e = a, \quad i = 1, \ldots, k,$$

as well as the equivariance condition

$$\sigma(a) \circ_{\sigma_i} \tau(b) = \iota_i(\sigma, \tau)(a \circ_i b),$$

where $\iota_i(\sigma, \tau) \in S_{k+l-1}$ is defined by letting τ permute the set

$$E_i = \{i, i+1, \ldots, i+l-1\}$$

of cardinality l and then by letting σ permute the set $\{1, \ldots, i-1, E_i, i+l, \ldots, k+l-1\}$ of cardinality k. The *global composition* is defined by

$$\gamma : \mathcal{P}_n \otimes \mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_n} \to \mathcal{P}_{k_1+\cdots+k_n},$$
$$(a, b_1, \ldots, b_n) \mapsto (\ldots((a \circ_n b_n) \circ_{n-1} b_{n-1}) \ldots) \circ_1 b_1.$$

The *free $\mathcal{P}$-algebra with one generator* is given by

$$F_{\mathcal{P}} : \bigoplus_{n\geq 1} \mathcal{P}_n / S_n.$$

The sum of the partial compositions yields a right pre-Lie algebra structure on $F_{\mathcal{P}}^+ := \bigoplus_{n\geq 2} \mathcal{P}_n / S_n$, namely

$$\bar{a} \lhd \bar{b} := \sum_{i=1}^{k} \overline{a \circ_i b}.$$

Following F. Chapoton [8] one can consider the pro-unipotent group $G_{\mathcal{P}}^e$ associated with the completion of the pre-Lie algebra $F_{\mathcal{P}}^+$ for the filtration induced by the grading. More precisely, Chapoton's group $G_{\mathcal{P}}$ is given by the elements $g \in \widehat{F_{\mathcal{P}}}$ such that $g_1 \neq 0$, whereas $G_{\mathcal{P}}^e$ is the subgroup of $G_{\mathcal{P}}$ formed by elements g such that $g_1 = e$.

Any element $a \in \mathcal{P}_n$ gives rise through γ to an n-ary operation $\omega_a : F_{\mathcal{P}}^{\otimes n} \to F_{\mathcal{P}}$, and for any $x, y_1, \ldots, y_n \in F_{\mathcal{P}}^+$ we have [30]:[2]

$$\omega_a(y_1, \ldots, y_n) \lhd x = \sum_{j=1}^{n} \omega_a(y_1, \ldots, y_j \lhd x, \ldots, y_n). \tag{4}$$

[2] We thank Muriel Livernet for having brought this point to our attention.

2.2 The pre-Lie operad. Recall that, for any operad $\mathcal{O}$, an $\mathcal{O}$-algebra is a vector space V together with a morphism of operads from $\mathcal{O}$ to the operad $\mathrm{Endop}(V)$, where $\mathrm{Endop}(V)_n := \mathcal{L}(V^{\otimes n}, V)$ with obvious symmetric group actions and compositions. In this sense, pre-Lie algebras are algebras over the *pre-Lie operad*, which has been described in detail by F. Chapoton and M. Livernet in [10] as follows: $\mathcal{PL}_n$ is the vector space of labelled rooted trees, and partial composition $s \circ_i t$ is given by summing all the possible ways of inserting the tree t inside the tree s at the vertex labelled by i.

The free right pre-Lie algebra with one generator is then given by the space $\mathcal{T} = \bigoplus_{n\geq 1} T_n$ of rooted trees, as quotienting with the symmetric group actions amounts to neglecting the labels. The pre-Lie operation $(s,t) \mapsto (s \leftarrow t)$ is given by the sum of the graftings of t on s at all vertices of s. As a consequence of (4) we have two pre-Lie operations on $\mathcal{T}' = \bigoplus_{n\geq 2} T_n$ which interact as follows [30]:

$$(s \leftarrow t) \lhd u = (s \lhd u) \leftarrow t + s \leftarrow (t \lhd u).$$

The first pre-Lie operation $\lhd$ comes from the fact that $\mathcal{PL}$ is an augmented operad, whereas the second pre-Lie operation $\leftarrow$ comes from the fact that $\mathcal{PL}$ is the pre-Lie operad itself!

2.3 Two Hopf algebras of rooted forests. Let us consider the two commutative Hopf algebras $\mathcal{H}_{CK}$ and $\mathcal{H}$ associated respectively to the left pre-Lie algebras $(\mathcal{T}, \rightarrow)$ and $(\mathcal{T}', \rhd)$ by the functor described in Paragraph 1.3. The first is the *Connes–Kreimer Hopf algebra of rooted forests* [13]. Its coproduct is described in terms of admissible cuts [20], or alternatively as follows [31], Paragraph 3.2 and Section 7: the set U of vertices of a forest u is endowed with a partial order defined by $x \leq y$ if and only if there is a path from a root to y passing through x. Any subset W of the set of vertices U of u defines a *subforest* w of u in an obvious manner, i.e., by keeping the edges of u that link two elements of W. The coproduct is then defined by

$$\Delta_{\mathrm{CK}}(u) = \sum_{\substack{V \amalg W = U \\ W < V}} v \otimes w.$$

Here the notation $W < V$ means that $x < y$ for any vertex x of w and any vertex y of v such that x and y are comparable. The Hopf algebra $\mathcal{H}_{CK}$ is graded by the number of vertices.

The coproduct on the second Hopf algebra $\mathcal{H}$ is defined as follows: one identifies the unit of $S(\mathcal{T}')$ with the rooted tree $\bullet$. A *subforest* of a tree t is either the trivial forest $\bullet$, or a collection $(t_1, \ldots, t_n)$ of pairwise disjoint subtrees of t, each of them containing at least one edge. In particular two subtrees of a subforest cannot have any common vertex.

Let s be a subforest of a rooted tree t. Denote by t/s the tree obtained by contracting each connected component of s onto a vertex. We turn $\mathcal{H}$ into a bialgebra by defining a coproduct $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ on each tree $t \in \mathcal{T}'$ by

$$\Delta(t) = \sum_{s \subseteq t} s \otimes t/s,$$

where the sum runs over all possible subforests (including the unit $\bullet$ and the full subforest t). As usual we extend the coproduct Δ multiplicatively onto $S(\mathcal{T}')$. This makes $\mathcal{H} := \bigoplus_{n\geq 0} \mathcal{H}_n$ a connected graded Hopf algebra, where the grading is defined by the number of edges.

It turns out that $\mathcal{H}_{CK}$ is a left comodule coalgebra over $\mathcal{H}$ [5]. The coaction $\Phi : \mathcal{H}_{CK} \to \mathcal{H} \otimes \mathcal{H}_{CK}$ is the algebra morphism given by $\Phi(\mathbf{1}) = \bullet \otimes \mathbf{1}$ and $\phi(t) = \Delta_{\mathcal{H}}(t)$ for any nonempty tree t.

2.4 Feynman graphs. Feynman graphs, the cornerstones of perturbative quantum field theory, are organized in a connected graded Hopf algebra (as it has been pointed out by D. Kreimer, see [13]). More precisely, the free commutative algebra spanned by connected one-particle irreducible graphs[3] is endowed with a connected graded Hopf algebra structure, the coproduct of a graph being built up from (not necessarily connected but locally one-particle irreducible) subgraphs and the associated contracted graph, obtained by shrinking all connected components of the subgraph on a point (see [13], [29]). This Hopf algebra is right-sided, thus yielding a left pre-Lie algebra structure on connected one-particle irreducible graphs by the Loday–Ronco correspondence of Section 1.3. The pre-Lie product is obtained by summing all the possible ways to insert one graph inside another at different places.

3 Pre-Lie algebras of vector fields

3.1 Flat torsion-free connections. Let M be a differentiable manifold, and let ∇ be the covariant derivative operator associated to a connection on the tangent bundle $\mathrm{T}M$. The covariant derivative is a bilinear operator on vector fields (i.e., two sections of the tangent bundle): $(X, Y) \mapsto \nabla_X Y$ such that the following axioms hold:

$$\nabla_{fX} Y = f \nabla_X Y,$$
$$\nabla_X (fY) = f \nabla_X Y + (X \cdot f) Y \quad \text{(Leibniz rule)}.$$

The torsion of the connection τ is defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

and the curvature tensor is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

The connection is *flat* if the curvature R vanishes identically, and *torsion-free* if $\tau = 0$. The following crucial observation is an immediate consequence of (1) (see [21]):

[3] A connected graph is one-particle irreducible (1PI) if it remains connected when an edge is removed.

Proposition 3.1. *For any smooth manifold M endowed with a flat torsion-free connection ∇, the space $\chi(M)$ of vector fields is a left pre-Lie algebra, with pre-Lie product given by*

$$X \rhd Y := \nabla_X Y.$$

Note that on $M = \mathbb{R}^n$ endowed with its canonical flat torsion-free connection, the pre-Lie product is given by

$$(f_i \partial_i) \rhd (f_j \partial_j) = f_i (\partial_i f_j) \partial_j.$$

3.2 Pre-history of pre-Lie: the work of A. Cayley. As early as 1857, A. Cayley [6] discovered a link between rooted trees and vector fields on the manifold $\mathbb{R}^n$ endowed with its natural flat torsion-free connection. A vector field, which is written as

$$X = \sum_{i=1}^{n} f_i \partial_i,$$

is seen as an *operandator* since it mixes partial derivation operators ∂_i and operands (i.e., functions) f_i. He remarks (in our modern notation) that for two vector fields X and Y the operator $X \circ Y - \nabla_X Y$ is symmetric in X and Y and calls it the "algebraic product" $X \times Y$ of X and Y. It is not difficult to check that for any $f \in C^\infty$ we have

$$(X \times Y)(f) = X \cdot (Y \cdot f) - (\nabla_X Y) \cdot f = f''(X, Y),$$

where f'' is the second derivative of f. This still makes sense if f is replaced by a third vector field Z, the result being a vector field. Then for any rooted tree t with n vertices, each vertex v being decorated by a vector field X_v, he defines a new vector field $\mathcal{Y}(t)$. We can understand the construction of $\mathcal{Y}(t)$ by the following recursive procedure [24]: the decorated tree t is obtained by grafting all its branches t_k on the root r decorated by the vector field $X_r = \sum_{i=1}^n f_i \partial_i$, i.e., it is written $B_+^{X_r}(t_1, \ldots, t_k)$. Hence we set:

$$\mathcal{Y}(\bullet_{X_r}) = X_r, \quad \mathcal{Y}(t) = \sum_{i=1}^{n} f_i(t) \partial_i \quad \text{with } f_i(t) = f_i^{(k)}(\mathcal{Y}(t_1), \ldots, \mathcal{Y}(t_n)).$$

3.3 Relating two pre-Lie structures. Keeping the notations of the previous section, consider the left grafting $\to$ of two decorated trees, namely

$$s \to t = \sum_{v \text{ vertex of } t} s \to_i v.$$

Considering the Cayley map $t \mapsto \mathcal{Y}(t)$ from vector field-decorated rooted trees to vector fields, it can be shown, as a direct consequence of the Leibniz rule, that

$$\mathcal{Y}(s \to t) = \mathcal{Y}(s) \rhd \mathcal{Y}(t).$$

In other words, the Cayley map $\mathcal{Y}$ is a pre-Lie algebra morphism. Now consider the map d_X from rooted trees to vector field-decorated rooted trees, which decorates each vertex by X. It is obviously a pre-Lie algebra morphism, and $F_X := \mathcal{Y} \circ d_X$ is the unique pre-Lie algebra morphism which sends the one-vertex tree $\bullet$ to the vector field X. The uniqueness of course comes from the fact that the span of rooted trees with left grafting is the free left pre-Lie algebra with one generator ([10], [14]).

3.4 B-series, composition and substitution. B-series have been defined by E. Hairer and G. Wanner, following the pioneering work of J. Butcher [4] on Runge–Kutta methods for the numerical resolution of differential equations. Remarkably enough, rooted trees have been revealed to be an adequate tool not only for vector fields but also for the numerical approximation of their integral curves. J. Butcher discovered that the Runge–Kutta methods form a group (since then called the Butcher group), which is nothing but the character group of the Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ [2].

Consider any left pre-Lie algebra $(A, \rhd)$, introduce a fictitious unit $\mathbf{1}$ such that $\mathbf{1} \rhd a = a \rhd \mathbf{1} = a$ for any $a \in A$, and consider for any $a \in A$ the unique left pre-Lie algebra morphism $F_a : (\mathcal{T}, \to) \to (A, \rhd)$ such that $F_a(\bullet) = a$. A *B-series* is an element of $hA[[h]] \oplus k \cdot \mathbf{1}$ defined by

$$B(\alpha; a) := \alpha(\varnothing)\mathbf{1} + \sum_{s \in T} h^{v(s)} \frac{\alpha(s)}{\sigma(s)} F_a(s),$$

where α is any linear form on $\mathcal{T} \oplus k\varnothing$. It matches the usual notion of B-series [24] when A is the pre-Lie algebra of vector fields on $\mathbb{R}^n$ defined at the beginning of the section (it is also convenient to set $F_a(\varnothing) = \mathbf{1}$). In this case, the vector fields $F_a(t)$ for a tree t are differentiable maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ called *elementary differentials*. B-series can be composed coefficientwise, as series in the indeterminate h whose coefficients are maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The same definition with trees decorated by a set of colours $\mathcal{D}$ leads to straightforward generalizations. For example, P-series used in partitioned Runge–Kutta methods [24] correspond to bi-coloured trees.

A slightly different way of defining B-series is the following: consider the unique pre-Lie algebra morphism $\mathcal{F}_a : \mathcal{T} \to hA[h]$ such that $\mathcal{F}_a(\bullet) = ha$. It respects the gradings given by the number of vertices and the powers of h respectively, hence it extends to $\mathcal{F}_a : \widehat{\mathcal{T}} \to hA[[h]]$, where $\widehat{\mathcal{T}}$ is the completion of $\mathcal{T}$ with respect to the grading. We further extend it to the empty tree by setting $\mathcal{F}_a(\varnothing) = \mathbf{1}$. We have then

$$B(\alpha; a) = \mathcal{F}_a \circ \tilde{\delta}^{-1}(\alpha),$$

where $\tilde{\delta}$ is the isomorphism from $\widehat{\mathcal{T}} \oplus k\varnothing$ to $(\mathcal{T} \oplus k\varnothing)^*$ given by the normalized dual basis.

We restrict ourselves to B-series $B(\alpha; a)$ with $\alpha(\varnothing) = 1$. Such α's are in one-to-one correspondence with characters of the algebra of forests (which is the underlying algebra of $\mathcal{H}_{CK}$) by setting

$$\alpha(t_1 \ldots t_k) := \alpha(t_1) \ldots \alpha(t_k).$$

The Hairer–Wanner theorem [24], Theorem III.1.10, says that composition of B-series corresponds to the convolution product of characters of $\mathcal{H}_{CK}$, namely

$$B(\beta; a) \circ B(\alpha; a) = B(\alpha * \beta, a),$$

where linear forms α, β on $\mathcal{T} \oplus k\varnothing$ and their character counterparts are identified modulo the above correspondence.

Let us now turn to substitution, following [12]. The idea is to replace the vector field a in a B-series $B(\beta; a)$ by another vector field $\tilde{a}$ which expresses itself as a B-series, i.e., $\tilde{a} = h^{-1} B(\alpha; a)$ where α is now a linear form on $\mathcal{T} \oplus k\varnothing$ such that $\alpha(\varnothing) = 0$. We suppose here moreover that $\alpha(\bullet) = 1$. Such α's are in one-to-one correspondence with characters of $\mathcal{H}$. The following proposition is proved in [5].

Proposition 3.2. *For any linear forms α, β on $\mathcal{T}$ with $\alpha(\bullet = 1)$, we have*

$$B(\beta; \tfrac{1}{h} B(\alpha; a)) = B(\alpha \star \beta; a),$$

where α is multiplicatively extended to forests, β is seen as an infinitesimal character of $\mathcal{H}_{CK}$ and where $\star$ is the dualization of the left coaction of $\mathcal{H}$ on $\mathcal{H}_{CK}$.

The condition $\alpha(\bullet) = 1$ is in fact dropped in [5], Proposition 15: the price to pay is that one has to replace the Hopf algebra $\mathcal{H}$ by a non-connected bialgebra $\widetilde{\mathcal{H}} = S(\mathcal{T})$ with a suitable coproduct, such that $\mathcal{H}$ is obtained as the quotient $\widetilde{\mathcal{H}}/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by $\bullet - \mathbf{1}$. The substitution product $\star$ then coincides with the one considered in [12] via natural identifications.

4 Links with other algebraic structures

A *dendriform algebra* [26] over the field k is a k-vector space A endowed with two bilinear operations, denoted $\prec$ and $\succ$ and called right and left products, respectively, subject to the three axioms below:

$$\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c),\\
(a \succ b) \prec c &= a \succ (b \prec c),\\
a \succ (b \succ c) &= (a \prec b + a \succ b) \succ c.
\end{aligned}$$

One readily verifies that these relations yield associativity for the product

$$a * b := a \prec b + a \succ b.$$

However, at the same time the dendriform relations imply that the bilinear products $\rhd$ and $\lhd$ defined by

$$a \rhd b := a \succ b - b \prec a, \quad a \lhd b := a \prec b - b \succ a,$$

are left pre-Lie and right pre-Lie, respectively. The associative operation $*$ and the pre-Lie operations $\rhd$, $\lhd$ all define the same Lie bracket:

$$[\![a,b]\!] := a * b - b * a = a \rhd b - b \rhd a = a \lhd b - b \lhd a.$$

We stress here that in the commutative case (commutative dendriform algebras are also called *Zinbiel algebras* [25], [26]) the left and right operations are further required to be identified so that $a \succ b = b \prec a$. In this case both pre-Lie products vanish. A natural example of a commutative dendriform algebra is given by the shuffle algebra in terms of half-shuffles [32]. Any associative algebra A equipped with a linear integral-like map $I : A \to A$ satisfying the integration by parts rule also gives a dendriform algebra, when $a \prec b := aI(b)$ and $a \succ b := I(a)b$. The left pre-Lie product is then given by $a \rhd b = [I(a), b]$. This construction can be also done with twisted versions of dendriform algebras, encompassing operators like the Jackson integral I_q [17].

Returning to ordinary dendriform algebras, observe that

$$a * b + b \rhd a = a \succ b + b \succ a.$$

This identity generalizes to any number of elements, expressing the symmetrization of $(\ldots((a_1 \succ a_2) \succ a_3)\ldots) \succ a_n$ in terms of the associative product and the left pre-Lie product [19].

A *brace algebra* is a vector space A together with a family of n-ary operations $(n \geq 2)$

$$A^{\otimes n} \to A, \quad a_1 \otimes \cdots \otimes a_n \mapsto \langle a_1, \ldots, a_{n-1}; a_n \rangle,$$

subject to the relations

$$\langle a_1, \ldots, a_m; \langle b_1, \ldots, b_n; c \rangle\rangle = \sum \langle A_0, \langle A_1; b_1 \rangle, \ldots, A_{2n-2}, \langle A_{2n-1}; b_n \rangle, A_{2n}; c \rangle,$$

where the sum runs over the partitions of the ordered set $\{a_1, \ldots, a_m\}$ into (possibly empty) consecutive intervals $A_0 \sqcup \cdots \sqcup A_{2n}$. The symbol $\langle b \rangle$ is identified with b for any $b \in A$. A brace algebra is *symmetric* if $\langle a_1, \ldots, a_{n-1}; a_n \rangle$ is symmetric in $a_1, \ldots, a_{n-1}$. If A is a symmetric brace algebra, then the product $(x, y) \mapsto x \rhd y := \langle x; y \rangle$ is left pre-Lie, and the other brace operations are given terms of it [23], so that the notions of symmetric brace algebra and pre-Lie algebra are equivalent.

Brace algebras are particular instances of *multibrace algebras*. For an account of these structures, see [27].

Acknowledgement. The author is grateful to the Erwin Schrödinger Institute (Wien) for warm hospitality during the writing of this article.

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Divergent multiple sums and integrals with constraints: a comparative study

Sylvie Paycha

Contents

Introduction

Assigning a finite value to $1 + 1 + \cdots + 1 + \cdots$, which can be done by means of the zeta function by declaring this is $\zeta(0)$, is a special instance of a more general issue, namely counting integral points on a cone, here the cone $\mathbb{R}_+$. This amounts to evaluating divergent discrete sums of functions (here the constant function identically equal to one) over integral points while imposing conical constraints on the arguments. Multiple zeta functions evaluated at zero count integral points on cones of the type $0 < x_k < x_{k-1} < \cdots < x_1$. Our first goal is to extend known techniques, which make sense of divergent multiple zeta functions [18], [27] and sums of polynomials at integral points of cones [4] to more general divergent sums of tensor products of symbols at integral points of cones. To do so one first needs to investigate the meromorphic behaviour of multiple discrete sums of symbols with conical constraints and to evaluate them at poles. The meromorphic behaviour of multiple zeta functions was studied in e.g. [1], [14], [40]; we show how a similar meromorphic behaviour holds for discrete sums on more general cones.

Our second goal is to describe and compare this meromorphic behaviour with that of multiple integrals of symbols with linear constraints studied in [30].

Multiple integrals of symbols with linear constraints generalise Feynman integrals in the absence of external momenta. Meromorphicity of Feynman integrals derived by means of dimensional or analytic regularisation procedures was previously investigated by various authors from different viewpoints, see e.g. [37], [10], [13], [9], [5], [6].

Whether considering multiple integrals with linear constraints or multiple sums with conical constraints, in the two cases the poles lie on discrete sets of hyperplanes and the hyperplanes passing through zero are all of the form $z_{i_1} + \cdots + z_{i_j} = 0$, where the z_i's are the complex parameters entering the regularisation. In view of this same meromorphic behaviour around zero, a similar renormalisation procedure using Speer's [37] generalised evaluators can be implemented for both sums and integrals with constraints. Evaluators are characters on algebras of meromorphic functions with linear pole structure, equipped with the tensor product. The multiplicativity property of evaluators on tensor products ensures that the corresponding renormalised sums and integrals factorise over disjoint sets of constraints.

The main goal of this article is therefore to describe the analogies in the pole structure and the mechanisms that underlie the pole structure of integrals with linear constraints, inspired from those arising from Feynman integrals on the one hand and of discrete sums on cones such as multiple zeta functions on the other hand.[1] We indeed draw a close parallel between two types of objects, namely

(1) multiple sums with conical constraints

$$\sum_{\mathcal{C}_A\cap\mathbb{Z}^k}\sigma_1\otimes\cdots\otimes\sigma_k := \sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{Z}^k}\sigma_1(x_1)\dots\sigma_k(x_k),$$

where $\mathcal{C}_A := \sum_{j=1}^k \mathbb{R}_+ v_j$ is an (open) cone generated by non-zero vectors $v_j = \sum_{i=1}^k a_{ij}e_i$, $j=1,\dots,J$, for some matrix $A=(a_{ij})_{i\in\{1,\dots,k\},j\in\{1,\dots,J\}}$, $\{e_1,\dots,e_k\}$ being the canonical orthonormal basis of $\mathbb{R}^k$, the σ_i, $i=1,\dots,k$ being classical symbols on $\mathbb{R}$,

(2) multiple integrals with linear constraints

$$\begin{aligned}\int_{(\mathbb{R}^n)^L}&(\sigma_1\otimes\cdots\otimes\sigma_I)\circ A\\ &:= \int_{\mathbb{R}^n} dk_1\dots\int_{\mathbb{R}^n} dk_L\,\Big[\sigma_1\Big(\sum_{l=1}^L a_{1l}\,k_l\Big)\dots\sigma_I\Big(\sum_{l=1}^L a_{Il}\,k_l\Big)\Big],\end{aligned}\tag{1}$$

given by a matrix $A=(a_{il})_{i\in\{1,\dots,I\},l\in\{1,\dots,L\}}$ of rank L with real coefficients and where the $\sigma_i: x\mapsto \tau_i(|x|)$, $i=1,\dots,I$ are radial[2] classical symbols on $\mathbb{R}^n$.

Multiple sums with conical constraints generalise multiple zeta functions. Indeed, choosing $J=k$ and setting $v_j=e_1+\cdots+e_j$, $\sigma_i(x)=\chi(x)x^{-s_i}$ for complex numbers s_i with sufficiently large real part, we have

$$\sum_{\mathcal{C}_A\cap\mathbb{Z}^k}\sigma_1\otimes\cdots\otimes\sigma_k = \sum_{1<n_k<\cdots<n_1} n_1^{-s_1}\dots n_k^{-s_k},$$

where χ is some smooth cut-off function which vanishes around zero and is identically one outside the unit interval. Here, the matrix A is given by $a_{ij}=1$ if $i\le j$ and $a_{ij}=0$ otherwise. For this choice of matrix A the sum $\sum_{\mathcal{C}_A\cap\mathbb{Z}^k}\sigma_1\otimes\cdots\otimes\sigma_k$ can also be written (note the analogy with (1))

$$\sum_{\vec{n}\in\mathbb{N}^J}(\sigma_1\otimes\cdots\otimes\sigma_k)\circ A := \sum_{\mathbb{N}^J}\Big[\Big(\sum_{j=1}^J a_{1l}\,n_l\Big)^{-s_1}\dots\Big(\sum_{j=1}^J a_{kj}n_j\Big)^{-s_k}\Big],$$

which corresponds to a particular instance of Shintani multiple zeta functions [25] associated with a matrix A with non-negative integer coefficients and no line of zeroes. The latter have a similar meromorphic behaviour to that of sums with conical constraints

[1] How Feynman diagrams (or equivalently Feynman graphs, see p. 2 in this volume) give rise to multiple zeta values is another issue which lies well beyond the scope of this article.

[2] This assumption, which we make for technical reasons, is probably not necessary, but at this stage we are not able to do without it.

studied here since they turn out to be linear combinations of sums on cones associated with bases of vectors extracted from the family of column vectors of A.[3]

Multiple integrals with linear constraints are similar to the ones arising from Feynman diagrams in the absence of external momenta; for example, choosing $\alpha_i, i = 1, 2, 3$ large enough, $n = 4$, $I = 3$, $\sigma_i(k) = \frac{1}{(|k|^2+1)^{\alpha_i}}$ we have

$$\begin{aligned}&\int_{(\mathbb{R}^4)^2} (\sigma_1 \otimes \sigma_2 \otimes \sigma_3) \circ A \\ &\quad = \int_{\mathbb{R}^4} dk_1 \int_{\mathbb{R}^r} dk_2 \, \frac{1}{(|k_1 + k_2|^2 + 1)^{\alpha_1}} \, \frac{1}{(|k_1|^2 + 1)^{\alpha_2}} \, \frac{1}{|k_2|^2 + 1)^{\alpha_3}},\end{aligned} \tag{2}$$

with $A(k_1, k_2) = (k_1 + k_2, k_2, k_3)$.

A common feature is that the "integrands" of both multiple sums and integrals with constraints, involve tensor products of polyhomogeneous symbols. That the constraints can be described in terms of matrices is another common feature. It is therefore natural to expect a similar pole structure for the meromorphic extensions of sums with conical constraints and integrals with linear constraints. We explore this analogy and present a common renormalisation method for the two setups.

Physicists have developed very sophisticated computational methods to renormalise multiple integrals with affine constraints arising from Feynman diagrams. These methods known under BPHZ renormalisation methods were later interpreted in algebraic terms by Kreimer [23] and then by Connes and Kreimer [8] using Birkhoff–Hopf factorisation on the Hopf algebra of Feynman diagrams. More recently, these methods were applied in [18] and [27], to renormalise multiple zeta values via Birkhoff–Hopf factorisation on Hopf algebras equipped with a stuffle product and renormalised sums of polynomial on cones were built in [4] using yet different renormalisation methods.

An important ingredient in these approaches is the *meromorphicity* of some integrals and sums obtained from *perturbing the integrands holomorphically*, for example using dimensional or Riesz regularisation. We adopt a unified approach to handle both the multiple sums and the multiple integrals with constraints, which is inspired from techniques used in [37]. We are guided by the following general scheme leading to renormalised integrals or sums:

(1) We first use a holomorphic regularisation $\mathcal{R}$ which embeds a symbol σ in a holomorphic family $\sigma(z)$ of non constant affine order in z. This induces a holomorphic perturbation $\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \ldots, z_k)$ around $(z_1, \ldots, z_k) = 0$ of the integrand $\sigma_1 \otimes \cdots \otimes \sigma_k$ under the sum[4] and integral symbols. Dimensional regularisation in physics or Riesz regularisation in mathematics (also called modified dimensional regularisation in physics) provide natural examples of holomorphic regularisations.

[3]I thank Bin Zhang for drawing my attention to Shintani multiple zeta functions, which we hope to investigate elsewhere in relation with multiple sums with conical constraints.

[4] $\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \ldots, z_k) = \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)$ does the job for integrals. However, one needs a twisted version of $\widetilde{\mathcal{R}}$ for sums on cones if one wants to preserve the stuffle product, see [27].

(2) Using iterated Mellin transforms combined with iterated integration by parts, we build meromorphic extensions

$$
\begin{aligned}
(z_1, \ldots, z_k) &\mapsto \sum_{\mathcal{C}_A \cap \mathbb{Z}^k} \widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \ldots, z_k), \\
(z_1, \ldots, z_I) &\mapsto \int_{(\mathbb{R}^n)^L} (\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_I)(z_1, \ldots, z_I)) \circ A
\end{aligned}
\tag{3}
$$

of the corresponding sums and integrals with constraints (see Theorems 5.8 and 6.4). We show that the poles are located on a countable set of affine hyperplanes, with the ones passing through zero of the form $z_{j_1} + \cdots + z_{j_i} = 0$, for any subsets $\{j_1, \ldots, j_i\}$ of positive integers. This generalises known meromorphicity results for multiple zeta functions [1], [14], [40] for which the hyperplanes of poles containing zero are of the form $z_1 + \cdots + z_j = 0$, j varying in $\mathbb{N}$.

(3) By analytic continuation, these meromorphic extensions (3) factorise for disjoint sets of constraints (see Corollaries 5.10 and 6.8), a property that we refer to as the multiplicativity property. It is reminiscent of the locality principle in physics by which distant objects cannot have direct influence on one another since an object is influenced directly only by its immediate surroundings.

(4) On the grounds of the pole structure, we further observe that if the orders a_i of the symbols σ_i have non-integer valued partial sums, i.e., if $a_{j_1} + \cdots + a_{j_i} \notin \mathbb{Z}$ for any subset $\{j_1, \ldots, j_i\}$ of positive integers, then the meromorphic extensions (3) are holomorphic at $(z_1, \ldots, z_k) = 0$. This is reminiscent of the fact that on non-integer order symbols, regularised integrals in one variable coincide with the canonical integral; the latter is holomorphic at zero on holomorphic families approaching a non-integer order symbol at zero.

(5) The last step consists in picking an appropriate finite part at zero of the meromorphic extensions (3) while preserving the multiplicative property. The common pole structure around zero allows us to use the same generalised evaluators on certain classes of meromorphic functions in several variables (Theorem 7.1). Their multiplicativity property ensures that the corresponding renormalised multiple integrals and sums with constraints factorise for disjoint sets of constraints.

(6) In specific situations, Birkhoff–Hopf factorisation on appropriately chosen Hopf algebras offers an alternative renormalisation method. A first instance of such a situation arises when fixing a certain type of conical constraint in the sums, such as when choosing Chen cones $\mathcal{C} = \langle e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{k-1}, e_1 + \cdots + e_k \rangle$. Another instance arises when fixing a certain type of integrand in the integrals, such as tensor powers of a given symbol $\sigma(k) = (|k|^2 + m^2)^{-1}$ where $m \neq 0$ plays the role of a mass in physics. In the last part of the article, we discuss how, in further identifying the complex parameters $z_i = z$ used to describe the pole structure around zero, the meromorphic extensions (3) give rise to morphisms on a Hopf algebra with values in the algebra of meromorphic functions in one variable. In the first case, it is a Hopf

algebra on tensor products of symbols equipped with the stuffle product [27]. In the second case, it is a Hopf algebra of sets of column vectors with coproduct given by deconcatenation. Applying Birkhoff–Hopf factorisation to these morphisms leads to another set of renormalised values.

(7) Specialising to Chen cones $\mathcal{C} = \langle e_1, \dots, e_1 + \dots + e_k \rangle$ and symbols $\sigma_i(x) = \chi(x)x^{-s_i}$, $i = 1, \dots, k$, where χ is a smooth cut-off function around zero, these constructions lead to different renormalised multiple zeta values according to which renormalisation method one chooses.

This article therefore suggests a unified presentation of mechanisms that underlie the renormalisation of integrals and sums with constraints. It hopefully clarifies the role of polyhomogeneous symbols as well as the requirements needed on the constraints for meromorphic extensions to exist.

It raises many questions, some of which were also raised in [17] and [16], in particular how different renormalised values obtained from different regularisation procedures and different generalised evaluators compare and how renormalised values obtained via evaluators compare with those obtained via Birkhoff–Hopf factorisation.

Acknowledgements. Let me first address my thanks to Michèle Vergne for her valuable advice and comments on preliminary versions of this article; I very much benefited from numerous discussions with her around renormalisation issues. I am also very indebted to Daniel Bennequin for his constant encouragement, and his constructive comments while I was writing this article. Parts of the article were written during a stay at the Max Planck Institute in Bonn, to which I am very grateful for the hospitality. I thank Matilde Marcolli, Li Guo and Bin Zhang for many stimulating discussions during various stays at the Max Planck Institute. I also thank the Hausdorff Research Institute for Mathematics in Bonn, where parts of this article were written, for its hospitality and a stimulating atmosphere. Let me finally thank Arthur Greenspoon most warmly for his help in editing this contribution.

Incipit

Underlying the issues discussed here is the following linear algebraic question which serves as a connecting thread throughout the article.

Let V be a linear space over a field k and let $\mathcal{F}(V)$ be an algebra of k-valued functions on V equipped with the usual product of functions. The choice of $\mathcal{F}(V)$ needs to be specified according to the context. Here are some examples of interest to us.

Example 1. • $\mathcal{F}(V) = k(V^*)$, the polynomial algebra on V with coefficients in k,

- $V = \mathbb{C}$ and the algebra $\mathcal{F}(V) = \mathrm{Hol}_0(\mathbb{C})$ of germs of complex valued holomorphic functions at zero,

- $V = \mathbb{C}$ and the algebra $\mathcal{F}(V) = \mathrm{Mer}_0(\mathbb{C})$ of germs of complex valued meromorphic functions around zero,
- $V = \mathbb{R}^d$ and $\mathcal{F}(V) = \mathrm{CS}(\mathbb{R}^d)$, the algebra of classical symbols on $\mathbb{R}^d$ with constant coefficients to be introduced shortly.

A linear form $L_n \in (k^n)^*$ reads $L_n(x_1, \dots, x_n) = \sum_{j=1}^n a_j x_j$ for some $(a_1, \dots, a_n)$ in k^n and induces a linear map, which by abuse of notation we denote by the same symbol:

$$L_n : V^n \to V, \quad (v_1, \dots, v_n) \mapsto \sum_{j=1}^n a_j v_j .$$

For every positive integer n, we consider the following subset of $\mathcal{F}(V^n)$. Setting $\mathcal{F} := \mathcal{F}(V)$ to simplify notations we define the algebra

$$\mathcal{F}^{(n)} := \left\{ \prod_{i=1}^I f_i \circ L_n^i,\ f_i \in \mathcal{F},\ L_n^i \in (k^n)^*,\ I \subset \mathbb{N} \right\}.$$

Let

$$\mathcal{T}(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \mathcal{F}^{\otimes n}, \quad \mathcal{T}(\mathcal{F})_{\mathrm{lin}} := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}.$$

Remark 1. In order to simplify the presentation, we omit any topological considerations for the moment, but we will later consider topological algebras and the associated completed tensor product.

In some cases, $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$ coincides with $\mathcal{T}(\mathcal{F})$; in others it is strictly larger.

Example 2. • If $\mathcal{F} = k(V^*)$, then $\mathcal{T}(\mathcal{F})_{\mathrm{lin}} = \mathcal{T}(\mathcal{F})$.

- If $\mathcal{F} = \mathrm{Hol}_0(\mathbb{C})$, then $\mathcal{T}(\mathcal{F})_{\mathrm{lin}} = \mathcal{T}(\mathcal{F}) = \bigoplus_{n=0}^{\infty} \mathrm{Hol}_0(\mathbb{C}^n)$.
- If $\mathcal{F} = \mathrm{Mer}_0(\mathbb{C})$, then $\mathcal{F}^{(n)}$ corresponds to germs of meromorphic functions at zero in n variables $(z_1, \dots, z_n)$ with linear poles $L_n(z_1, \dots, z_n) = 0$, $L_n \in (k^n)^*$. It differs from the tensor product $(\mathrm{Mer}_0(\mathbb{C}))^{\otimes n}$, which consists of products of germs of meromorphic functions at zero in one variable. For example, $(z_1, z_2) \mapsto \frac{z_1}{z_1 + z_2}$ lies in $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$ but not in $\mathcal{T}(\mathcal{F})$.
- If $\mathcal{F} = \mathrm{CS}(\mathbb{R}^d)$, again we have $\mathcal{T}(\mathcal{F})_{\mathrm{lin}} \neq \mathcal{T}(\mathcal{F})$, as can be seen from the map

  $$(k_1, k_2) \mapsto \frac{1}{(|k_1 + k_2|^2 + 1)} \frac{1}{(|k_1|^2 + 1)} \frac{1}{|k_2|^2 + 1)}$$

  considered in the introduction (see (2)) since it lies in $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$ but not in $\mathcal{T}(\mathcal{F})$.

The linear group $\mathrm{GL}_\infty(k)$ acts on $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$ on the left by

$$\mathrm{GL}_\infty(k) \times \mathcal{T}(\mathcal{F})_{\mathrm{lin}} \to \mathcal{T}(\mathcal{F})_{\mathrm{lin}} :$$
$$(A_n, \prod_{i \in I} f_i \circ l_n^i) \in \mathrm{GL}_n(k) \times \mathcal{F}^{(n)} \mapsto \prod_{i \in I} f_i \circ L_n^i \circ A_n^{-1}.$$

The set $\mathcal{T}(\mathcal{F})_{\text{lin}}$ equipped with the product

$$\prod_{i\in I}(f_i \circ L_n^i) \bullet \prod_{i'\in I'}(f'_{i'} \circ L'^{i'}_{n'}) := \prod_{i\in I} f_i \circ (L_n^i \oplus 0_{n'}) \cdot \prod_{i'\in I'} f'_{i'} \circ (0_n \oplus (L')^{i'}_{n'})$$

is a graded k-algebra. It has a natural filtration given by $\mathcal{F}_n := \bigoplus_{i=0}^n \mathcal{F}^{(i)}$.

Let $\delta_i : k^n \to k$ be the linear form which sends the j-th vector in the canonical basis to 1 if $j = i$ and to zero otherwise. We have a natural injection of algebras:

$$i : \mathcal{F}^{\otimes n} \to \mathcal{F}^{(n)}, \quad f_1 \otimes \cdots \otimes f_n \mapsto \prod_{i=1}^n f_i \circ \delta_i.$$

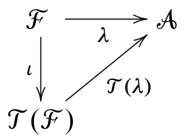
Combined with the identification $\mathcal{F} \simeq \mathcal{F}^{\otimes 1} \subset \mathcal{T}(\mathcal{F})$, this induces the following set of inclusions of algebras

$$\mathcal{F} \subset \mathcal{T}(\mathcal{F}) \subset \mathcal{T}(\mathcal{F})_{\text{lin}}.$$

Let $\mathcal{A}$ be another k-algebra. By the universal property of tensor algebras a linear map $\lambda : \mathcal{F} \to \mathcal{A}$ induces a unique algebra morphism

$$\mathcal{T}(\lambda) : \mathcal{T}(\mathcal{F}) \to \mathcal{A}, \quad (f_1 \otimes \cdots \otimes f_n) \mapsto \prod_{i=1}^n \lambda(f_i),$$

leading to the diagram



where ι is the canonical inclusion.

If $\mathcal{T}(F)_{\text{lin}} = \mathcal{T}(\mathcal{F})$, the map λ trivially extends to a morphism $\mathcal{T}(\lambda)_{\text{lin}} = \mathcal{T}(\lambda)$. Here are examples that illustrate this situation.

Example 3. Evaluation at zero, $\text{ev}_0 : f \mapsto f(0)$, on polynomials in one variable or on germs of complex valued holomorphic functions around zero in one variable, canonically extends to evaluation at zero on polynomials in several variables or on germs of holomorphic functions in several variables around zero.

The following result describes extensions in the more general case $\mathcal{T}(\mathcal{F})_{\text{lin}} \neq \mathcal{T}(\mathcal{F})$.

Proposition 1. *Let $\lambda : \mathcal{F} \to k$ be a character. If for every $f = \prod_{i\in I}(f_i \circ L_n^i)$ in $\mathcal{T}(\mathcal{F})_{\text{lin}}$ and for every $j \in \{1, \dots, n\}$ the map*

$$\lambda_j(f) : (v_1, \dots, \hat{v_j}, \dots v_n) \mapsto \lambda(f(v_1, \dots, \hat{v_j}, \dots, v_n)) \tag{4}$$

extends a map in $\mathcal{T}(\mathcal{F})_{\text{lin}}$, then λ induces characters $\lambda_{\to}^{\text{ren}} : \mathcal{T}(\mathcal{F})_{\text{lin}:} \to k$ and $\lambda_{\leftarrow}^{\text{ren}} : \mathcal{T}(\mathcal{F})_{\text{lin}:} \to k$, respectively, defined by

$$\lambda_{\to}^{\text{ren}} := \lambda_1 \circ \cdots \circ \lambda_n \quad \textit{and} \quad \lambda_{\leftarrow}^{\text{ren}} := \lambda_n \circ \cdots \circ \lambda_1,$$

as well as their symmetrised version

$$\lambda^{\mathrm{ren,sym}}\Big(\prod_{i\in I} f_i\circ L_n^i\Big)=\frac{1}{n!}\sum_{\tau\in\Sigma_n}\lambda^{\overset{\mathrm{ren}}{\to}}\Big(\prod_{i\in I_n} f_i\circ L_n^i\circ\tau^*\Big),$$

where τ^* *stands for the linear map* $(v_1,\dots,v_n)\mapsto(v_{\tau(1)},\dots,v_{\tau(n)})$.

Example 4. Consider the function $f(z_1,z_2)=\frac{1}{z_1+z_2}$ in $\mathcal{T}\,\mathrm{Mer}_0(\mathbb{C})_{\mathrm{lin}}$ and $\lambda:=\mathrm{ev}_0^{\mathrm{reg}}$. Then $\lambda_2(f)(z_1)=\frac{1}{z_1}$ for $z_1\neq 0$ and $\lambda_2(f)(0)=\lambda_2\circ\lambda_1(f)=0$ so that $\lambda_2(f)$ extends the function $z_1\mapsto\frac{1}{z_1}$.

Proof. Let us set $f:=\prod_{i\in I_{n+1}} f_i\circ L_{n+1}^i$. We first observe that the map $\lambda_n^{\overset{\mathrm{ren}}{\to}}$ is well defined by induction on n, for the assumption of the proposition implies that

$$(v_1,\dots,v_n)\mapsto\lambda_{n+1}(f(v_1,\dots,v_n,\cdot))$$

extends a map in $\mathcal{F}$ and we have

$$\begin{aligned}\lambda_{n+1}^{\overset{\mathrm{ren}}{\to}}(f)&=\lambda_1(\dots(\lambda_{n+1}(f))\dots)\\&=\lambda_n^{\overset{\mathrm{ren}}{\to}}((v_1,\dots,v_n)\mapsto\lambda_{n+1}(f(v_1,\dots,v_n,\cdot)))\\&=\lambda_n^{\overset{\mathrm{ren}}{\to}}\circ\lambda_{n+1}(f).\end{aligned}$$

Similar observations hold for $\lambda^{\overset{\mathrm{ren}}{\leftarrow}}$. By their very construction, the maps $\lambda^{\overset{\mathrm{ren}}{\to}}$ and $\lambda^{\overset{\mathrm{ren}}{\leftarrow}}$ are compatible with the filtration by n. These maps are moreover compatible with the product $\bullet$. For $\lambda^{\overset{\mathrm{ren}}{\to}}$ this follows from

$$\begin{aligned}&\lambda^{\overset{\mathrm{ren}}{\to}}\Big(\prod_{i\in I} f_i\circ L_n^i\bullet\textstyle\prod_{i'\in I'} f'_{i'}\circ(L')_{n'}^{i'}\Big)\\&\quad=\lambda_{n+n'}\circ\dots\circ\lambda_{n'+1}\Big(\prod_{i\in I} f_i\circ L_n^i\Big(\lambda_{n'}\circ\dots\circ\lambda_1\Big(\prod_{i'\in I'} f'_{i'}\circ(L')_{n'}^{i'}\Big)\Big)\Big)\\&\quad=\lambda^{\overset{\mathrm{ren}}{\to}}\Big(\prod_{i\in I} f_i\circ L_n^i\Big)\lambda^{\overset{\mathrm{ren}}{\to}}\Big(\prod_{i'\in I'} f'_{i'}\circ(L')_{n'}^{i'}\Big).\end{aligned}$$

A similar proof holds for $\lambda^{\overset{\mathrm{ren}}{\leftarrow}}$. The multiplicativity of $\lambda^{\mathrm{ren,sym}}$ then follows. □

Note that $\lambda^{\overset{\mathrm{ren}}{\leftarrow}}$ can be derived from $\lambda^{\overset{\mathrm{ren}}{\to}}$ using a permutation matrix which reverses the indices. More generally, from an extension $\mathcal{T}(\lambda)_{\mathrm{lin}}:\mathcal{T}(\mathcal{F})_{\mathrm{lin}}\to k$, one can build other characters on $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$ using the action of $\mathrm{GL}_\infty(k)$ on $\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$:

$$\mathcal{T}(\lambda)_{\mathrm{lin}}^A:\mathcal{T}(\mathcal{F})_{\mathrm{lin}}\to k,\quad \prod_{i\in I} f_i\circ L_n^i\mapsto\mathcal{T}(\lambda)_{\mathrm{lin}}\Big(\prod_{i\in I} f_i\circ(L_n^i\circ A_n^{-1})\Big),$$

with $A_n\in\mathrm{GL}_n(k)$.

The examples mentioned above for which $\mathcal{T}(\mathcal{F})_{\rm lin} \neq \mathcal{T}(\mathcal{F})$ lie at the heart of these notes. We build extensions $\mathcal{T}(\lambda)_{\rm lin}$ for the following linear forms:

Example 5. (1) Regularised evaluators at zero on germs $\mathcal{F} = \mathrm{Mer}_0(\mathbb{C})$ of complex valued meromorphic functions around zero, in which case we can choose $\mathcal{T}(\lambda)_{\rm lin} = \lambda^{\overset{\rm ren}{\to}}$ which corresponds to a generalised evaluator at zero (see Proposition 1.11 in Section 1).

(2) Regularised integrals (see Section 3) on the algebra $\mathcal{F} = \mathrm{CS}(\mathbb{R}^d)$ of polyhomogeneous symbols on $\mathbb{R}^d$. Whether the assumption of the proposition holds is still unclear to us, namely whether the map defined by regularised integrals in one of the variables falls in the appropriate class $\mathcal{T}(\mathcal{F})_{\rm lin}$ as a map in the remaining variables. Instead, we use generalised evaluators or Birkhoff–Hopf factorisation (see Section 6 and Section 7).

(3) Regularised discrete sums (see Section 3) on the algebra $\mathcal{F} = \mathrm{CS}(\mathbb{R})$ of polyhomogeneous symbols on $\mathbb{R}$. Here again, it is unclear to us whether the assumption of the proposition is satisfied. Again, we use generalised evaluators or Birkhoff–Hopf factorisation (see Sections 5 and Section 7).

To our knowledge, the classification of extensions $\mathcal{T}(\lambda)_{\rm lin}$, which is closely related to the characterisation of $\mathcal{T}(\mathcal{F})_{\rm lin}$ by some universal property, remains an open question.

1 Evaluating meromorphic functions at poles: generalised evaluators

Evaluating at zero meromorphic functions in one variable that present a pole at that point is a common task among physicists. For meromorphic functions in one variable, the minimal subtraction scheme used by physicists is one way to extract a finite part at zero, which coincides with ordinary evaluation at zero in the absence of poles. It yields a regularised evaluator at zero which extends ordinary evaluation at that point. We classify regularised evaluators (Theorem 1.4) via a description of all possible linear extensions of the ordinary evaluation at zero to germs of meromorphic functions at zero. For meromorphic functions in several variables, the procedure is more involved; we restrict to meromorphic functions with linear poles which arise both in number theory and quantum field theory. Iterating regularised evaluation in one variable is one way to evaluate at zero meromorphic functions in several variables with linear poles; we conjecture that this is the only possible way and call such a procedure a generalised evaluator at zero induced by a regularised evaluator in one variable. The linear group $\mathrm{GL}_\infty(\mathbb{C})$ acts on these generalised evaluators via a linear change of variable; our conjecture would imply that linear changes of variable are the only degrees of freedom left in evaluating at zero a meromorphic function with linear poles, once the evaluator in one variable is fixed.

1.1 Minimal subtraction scheme. Let $\mathrm{Mer}_0^k(\mathbb{C})$ be the germs of meromorphic functions at zero[5] with poles at zero of order no larger than k, and let $\mathrm{Hol}_0(\mathbb{C})$ be the germs of holomorphic functions at zero.

Inspired by Speer [37] we introduce the notion of regularised evaluator.

Definition 1.1. We call *regularised evaluator* at zero any linear form $\lambda\colon \mathrm{Mer}_0(\mathbb{C}) \to \mathbb{C}$ which extends the following evaluator at zero on holomorphic functions at zero:

$$\mathrm{ev}_0\colon \mathrm{Hol}_0(\mathbb{C}) \to \mathbb{C}, \quad f \mapsto f(0).$$

One way of building a regularised evaluator at zero is to compose the evaluator with an appropriate Rota–Baxter operator.

Definition 1.2. A linear operator $R\colon \mathcal{A} \to \mathcal{A}$ on a (not necessarily associative) algebra $\mathcal{A}$ over a field F is called Rota–Baxter[6] of weight $\lambda \in F$ if it satisfies the following *Rota–Baxter relation*:

$$R(a)R(b) = R(R(a)b) + R(aR(b)) + \lambda R(ab).$$

Remark 1.3. If $\lambda \neq 0$, replacing R by $\lambda^{-1} R$ gives rise to a Rota–Baxter operator of weight 1.

If $f(z) = \sum_{i=k}^{\infty} a_i z^i$ we set $\mathrm{Res}_0^j(f) := a_{-j}$, called the j-th residue of f at zero. Let $\mathrm{Mer}_0(\mathbb{C}) := \cup_{k=0}^{\infty} \mathrm{Mer}_0^k(\mathbb{C})$. The projection map

$$R_+\colon \mathrm{Mer}_0(\mathbb{C}) \to \mathrm{Hol}_0(\mathbb{C}),\ f \mapsto \Big(z \mapsto f(z) - \sum_{j=1}^{k} \frac{\mathrm{Res}_0^j(f)}{z^j}\Big),\ \text{if } f \in \mathrm{Mer}_0^k(\mathbb{C}),$$

corresponds to what physicists call the *minimal subtraction scheme*. Whereas $R_+(f)$ corresponds to the holomorphic part, $R_-(f) = (1 - R_+)(f)$ corresponds to the "pole part".

The contribution $R_+(fg)$ to the holomorphic part of the product of two meromorphic functions f and g differs from the product $R_+(f)R_+(g)$ of the holomorphic parts of f and g by contributions of the poles through $R_-(f)$ and $R_-(g)$:

$$R_+(fg) = R_+(f)R_+(g) + R_+(fR_-(g)) + R_+(gR_-(f)).$$

The maps R_+ and R_- are both Rota–Baxter maps of weight -1 on $\mathrm{Mer}_0(\mathbb{C})$, i.e.,

$$R_+(a)R_+(b) = R_+(R_+(a)b) + R_+(aR_+(b)) - R_+(ab)$$

and similarly for R_-.

[5] I.e., equivalence classes of meromorphic functions defined on a neighborhood of zero for the equivalence relation $f \sim g$ if f and g coincide on some open neighborhood of zero.

[6] The concept of Rota–Baxter operator was actually introduced by Glen Baxter in [2] and further developed by Gian-Carlo Rota in [34].

Combining the evaluation at zero with the map R_+ provides a first regularised evaluator on $\mathrm{Mer}_0^k(\mathbb{C})$ at zero. The map

$$\mathrm{ev}_0^{\mathrm{reg}}\colon \mathrm{Mer}_0^k(\mathbb{C}) \to \mathbb{C}, \quad f \mapsto \mathrm{ev}_0 \circ R_+(f), \tag{5}$$

is a linear form that extends the ordinary evaluator $\mathrm{ev}_0(f) = f(0)$ defined on the space $\mathrm{Hol}_0(\mathbb{C})$ of holomorphic functions at zero. The following result provides a classification of regularised evaluators. The parameter μ that arises here is related to the renormalisation group parameter in quantum field theory.

Theorem 1.4. *Regularised evaluators at zero on* $\mathrm{Mer}_0^k(\mathbb{C})$ *are of the form*

$$\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \sum_{j=1}^{k} \mu_j \mathrm{Res}_0^j \tag{6}$$

for some constants $\mu_1, \dots, \mu_k$.

In particular, regularised evaluators at zero on $\mathrm{Mer}_0^1(\mathbb{C})$ *are of the form*

$$\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \mu \mathrm{Res}_0$$

for some constant μ.

Proof. A linear form λ which extends ev_0 coincides with ev_0 on the range of R_+ and therefore fulfils the following identity: $\lambda \circ R_+ = \mathrm{ev}_0 \circ R_+ = \mathrm{ev}_0^{\mathrm{reg}}$. Thus, for any $f \in \mathrm{Mer}_0^k(\mathbb{C})$,

$$\lambda(f) = \lambda(R_+(f)) + \lambda(R_-(f)) = \mathrm{ev}_0^{\mathrm{reg}}(f) + \sum_{j=1}^{k} c_j \mathrm{Res}_0^j(f),$$

where we have set $c_j := \lambda(z^{-j})$. □

Corollary 1.5. *Let* $f = \frac{h}{z^k}$ *be a meromorphic function around zero with a pole of order* k *at zero, and let*

$$\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \sum_{j=1}^{k} \mu_j \mathrm{Res}_0^j$$

for some constants $\mu_1, \dots, \mu_k$.

The function f *evaluated at zero is an affine expression in the parameters* μ_j *with coefficients expressed in terms of the jets of* h *at zero of order* k.

Proof. This easily follows from the fact that $\mathrm{ev}_0^{\mathrm{reg}}(f) = \frac{h^{(k)}(0)}{k!}$ combined with the fact that $\mathrm{Res}_0^j(f) = \mathrm{ev}_0^{\mathrm{reg}}(z^j f) = \frac{h^{(k-j)}(0)}{(k-j)!}$. □

1.2 Generalised evaluators. Extracting a reasonable finite part from meromorphic functions in several variables is more delicate than it is for functions in one variable. Generalised evaluators used by physicists (see e.g. [37]) provide a procedure to extract "multiplicative" finite parts from meromorphic functions with nested hyperplanes of poles.

Such methods apply to subsets of the algebra $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$ with the notations of the Incipit. Let us first observe that an element of $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$ is a meromorphic function in say k variables with linear poles, i.e., a function $f:\mathbb{C}^k\mapsto\mathbb{C}$ such that $f\prod_{i\in I_k}(L_i^k)^{m_i^k}$ is holomorphic at zero for some linear forms L_i^k, $i\in I_k$, on $\mathbb{C}^k$.

We now fix a family $\{L_i^k\mid\mathbb{C}^k\to\mathbb{C},\, i\in I_k\}_{k\in\mathbb{N}}$ of sets of linear forms L_i^k where $I_k\subset\mathbb{N}$ or equivalently a family of hyperplane arrangements $\{L_i^k=0,\, i\in I_k\}$ each of which is taken in $V=\mathbb{C}^k$ [11] and consider the subset $\mathcal{S}=\bigcup_{k=1}^{\infty}\mathcal{S}_k$ of $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$ defined by

$$\begin{aligned}\mathcal{S}_k := \{f:\mathbb{C}^k\mapsto\mathbb{C},\ &\text{there exists } m_i^k\in\mathbb{N},\ i\in I_k,\ \text{such that}\\ &\text{the map } f\textstyle\prod_{i\in I_k}(L_i^k)^{m_i^k} \text{ is holomorphic around } \vec{z}=0\}.\end{aligned}\tag{7}$$

A function f in $\mathcal{S}_k$ reads $f=\frac{h}{\prod_{i\in I_k}(L_i^k)^{m_i^k}}$ with h holomorphic around zero. Since the forms $L_i^k, i\in I_k$ are not required to be independent, the set of linear forms $\{L_i^{k'}, i\in I_k\}$ can be arbitrarily enlarged by multiplying h and the denominator by an arbitrary linear form L in the variables $z_1,\dots,z_k$. Enlarging the sets of linear forms corresponding to the poles if necessary, we can make the following assumptions:

- *Nested pole structure*: if $k\le k'$ then the linear forms L_i^k with i varying in I_k lie among the set of linear forms $\{L_i^{k'},\, i\in I_{k'}\}$, where a function in the variables $z_1,\dots,z_k$ is viewed as a function in the variables $z_1,\dots,z_{k+1}$ which is constant in z_{k+1}.
- *Compatibility with the tensor product*:

$$L_i^k\otimes L_j^{k'}\subset\{L_n^{k+k'},\, n\in I_{k+k'}\}\quad\text{for all } (k,k')\in\mathbb{N}^2,\ i\in I_k,\ j\in I_{k'}.$$

Under this second assumption, $\mathcal{S}$ is a filtered algebra for the tensor product corresponding to the product $\bullet$ on $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$. Indeed, let $f=\frac{h}{\prod_{i\in I_k}(L_i^k)^{m_i^k}}\in\mathcal{S}_k$ and $f'=\frac{h'}{\prod_{i\in I_{k'}}(L_i^{k'})^{m_i^{k'}}}\in\mathcal{S}_{k'}$ for some holomorphic functions h on $\mathbb{C}^k$ and h' on $\mathbb{C}^{k'}$. Then $f\otimes f'=\frac{h\otimes h'}{\prod_{i\in I_k}(L_i^k)^{m_i^k}\otimes\prod_{i\in I_{k'}}(L_i^{k'})^{m_i^{k'}}}$ lies in $\mathcal{S}_{k+k'}$. We henceforth call algebras of the above type *filtered algebras of type* (7). In practice we often have $m_i^k=1$ (see examples below).

Example 1.6. The tensor algebra $\mathcal{T}\left(\mathrm{Mer}_0^1(\mathbb{C})\right)$ on the linear space $\mathrm{Mer}_0^1(\mathbb{C})$ of germs of meromorphic functions in one variable with a simple pole at zero yields a filtered algebra of type (7) with linear forms $L_i^k(z_1,\dots,z_k)=z_i$, poles of multiplicity 1 and parameter set $I_k=\{1,\dots,k\}$.

We now consider subsets of the larger algebra $\mathcal{T}(\mathrm{Mer}_0^1(\mathbb{C}))_{\mathrm{lin}}$ given by

$$\begin{aligned}\mathcal{A}_k := \{ f\colon \mathbb{C}^k \mapsto \mathbb{C} &\text{ such that the map}\\ &h\colon (z_1,\dots,z_k)\mapsto f(z_1,\dots,z_k)\big(\textstyle\prod_{j=1}^k(z_1+\dots+z_j)\big) \text{ is holomorphic}\\ &\text{around } \vec{z}=0 \text{ and } h \text{ does not vanish on } z_1+\dots+z_j=0,\ j\in\{1,\dots,k\}\}.\end{aligned}$$

They do not give rise to a filtered algebra of type (7) since the sets of linear forms $L_j^k(z_1,\dots,z_k):=z_1+\dots+z_j$ labelled by $I_k=\{1,\dots,k\}$ are not stable under tensor product.

Example 1.7. By contrast, the subsets of $\mathcal{T}(\mathrm{Mer}_0^1(\mathbb{C}))_{\mathrm{lin}}$ given by

$$\begin{aligned}\mathcal{B}_k := \{f\colon \mathbb{C}^k \mapsto \mathbb{C} &\text{ such that the map } h\colon (z_1,\dots,z_k)\mapsto f(z_1,\dots,z_k)\\ &\cdot \textstyle\prod_{\tau\in\Sigma_k}\big(\prod_{j=1}^k(z_{\tau(1)}+\dots+z_{\tau(j)})\big) \text{ is holomorphic around zero and}\\ &h \text{ does not vanish on } z_{\tau(1)}+\dots+z_{\tau(j)}=0,\ j\in\{1,\dots,k\},\ \tau\in\Sigma_k\}\end{aligned} \tag{8}$$

yield a filtered algebra $\mathcal{B}:=\bigcup_{k\in\mathbb{N}}\mathcal{B}_k$ of type (7) with linear forms

$$L_{j,\tau}^k(z_1,\dots,z_k):=z_{\tau(1)}+\dots+z_{\tau(j)}$$

with poles of multiplicity 1.

Inspired by physicists (see e.g. [37]), we give the following definition, which is actually less restrictive than Speer's definition since we have left out symmetry in the variables and a continuity assumption.

Definition 1.8 (see e.g. [37]). A *generalised evaluator* at zero on an algebra $\mathcal{S}$ of type (7) adapted to the filtration $\mathcal{S}=\bigcup_{k=1}^\infty \mathcal{S}_k$ on $\mathcal{S}$ is a character $\Lambda\colon \mathcal{S}\to\mathbb{C}$ compatible with the filtration which extends the ordinary evaluation at zero on holomorphic functions around zero.

In other words, a generalised evaluator at zero is a family of maps $\Lambda=\{\Lambda_k,\ k\in\mathbb{N}\}$, $\Lambda_k\colon \mathcal{S}_k\to\mathbb{C}$, such that

(1) Λ is linear,

(2) Λ coincides with the evaluation at 0 on holomorphic functions around 0,

(3) Λ is compatible with the filtration on $\mathcal{S}$, namely $\Lambda_{k+1}|_{\mathcal{S}_k}=\Lambda_k$,

(4) Λ is multiplicative on tensor products,

$$\Lambda_{k+k'}(f\otimes f')=\Lambda_k(f)\Lambda_{k'}(f'),$$

for any $f\in\mathcal{S}_k$ depending only on the first k variables $z_1,\dots,z_k$ and $f'\in\mathcal{S}_{k'}$ on the remaining k' variables $z_{k+1},\dots,z_{k+k'}$.

Example 1.9. Generalised evaluators Λ at zero on the tensor algebra $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))$ are entirely determined by regularised evaluators $\lambda := \Lambda_1$ at zero, for the multiplicativity on tensor products yields

$$\Lambda_k(f_1 \otimes \cdots \otimes f_k) = \prod_{i=1}^{k} \lambda(f_i) \quad \text{for all } f_i \in \mathrm{Mer}_0(\mathbb{C}).$$

Example 1.10. The linear form defined by

$$\mathrm{ev}_0^{\overset{\mathrm{ren}}{\rightarrow}}|_{\mathcal{S}_k} := \mathrm{ev}_{z_1=0}^{\mathrm{reg}} \circ \cdots \circ \mathrm{ev}_{z_k=0}^{\mathrm{reg}}$$

and the linear form defined by

$$\mathrm{ev}_0^{\overset{\mathrm{ren}}{\leftarrow}}|_{\mathcal{S}_k} := \mathrm{ev}_{z_k=0}^{\mathrm{reg}} \circ \cdots \circ \mathrm{ev}_{z_1=0}^{\mathrm{reg}}$$

yield generalised regularised evaluators at zero. Here the subscript $z_i = 0$ means that the regularised evaluator $\mathrm{ev}_0^{\mathrm{reg}}$ is implemented in the variable z_i.

These examples belong to the following class of generalised evaluators at zero. The subsequent result follows from Proposition 1.

Proposition 1.11. *Let $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$ be a filtered algebra of type (7) such that for any positive integer k*

$$\{L_i^{k+1}|_{z_j=0},\, i \in I_{k+1}\} \subset \{L_i^k,\, i \in I_k\} \quad \textit{for all } j \in \{1, \ldots, k+1\}. \tag{9}$$

Then, given a regularised evaluator λ on $\mathcal{S}_1$, the linear form

$$\lambda_k^{\overset{\mathrm{ren}}{\rightarrow}} := \lambda_{z_1} \circ \cdots \circ \lambda_{z_k}$$

on $\mathcal{S}_k$ yields a generalised evaluator $\lambda^{\overset{\mathrm{ren}}{\rightarrow}}$ on $\mathcal{S}$ called the ***generalised evaluator induced by $\boldsymbol{\lambda}$*** *adapted to the filtration on $\mathcal{S}$ and*

$$\lambda_k^{\overset{\mathrm{ren}}{\leftarrow}} := \lambda_{z_k} \circ \cdots \circ \lambda_{z_1}$$

on $\mathcal{S}_k$ yields a generalised evaluator $\lambda^{\overset{\mathrm{ren}}{\leftarrow}}$ on $\mathcal{S}$ called the ***reverse generalised evaluator induced by $\boldsymbol{\lambda}$*** *adapted to the filtration on $\mathcal{F}$.*

Here λ_{z_j} defined similarly to (4) amounts to implementing λ in the variable z_j.

Proof. Since $\mathcal{S} \subset \mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$, the same proof as in Proposition 1 applies; we need to check that the assumption of Proposition 1 is satisfied. Let us check that for any $j \in \{1, \ldots, k+1\}$, the map λ_j given by

$$(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots z_{k+1}) \mapsto \lambda\Big(\prod_{i \in I_{k+1}} (f_i \circ L_i)(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_{k+1})\Big)$$

extends a map in $\mathcal{S}_k$ We first observe that $\lambda := \mathrm{ev}_0^{\mathrm{reg}}$ has this property due to assumption (9). Given any $j \in \{1, \dots, k+1\}$, a function f in $\mathcal{S}_{k+1}$ reads

$$f(z_1, \dots, z_{k+1}) = \sum_{l_j=L_j}^{M_j} z_j^{l_j} \frac{h_{l_j}(z_1, \dots, z_{k+1})}{\prod_{i \in I_{k+1}^j} (L_i^{k+1})^{m_i^{k+1}}}$$

for some integers $L_j \leq M_j$ and where the h_{l_j}'s are holomorphic functions and $\{L_i^{k+1}, i \in I_{k+1}^j\} \subset \{L_i^{k+1}, i \in I_{k+1}\}$ linear forms, neither of which vanishes identically on the hyperplane $z_j = 0$. Implementing $\mathrm{ev}_{z_j=0}^{\mathrm{reg}}$ yields

$$\mathrm{ev}_{z_j=0}^{\mathrm{reg}}(f(z_1, \dots, z_{k+1})) = \frac{h_0(z_1, \dots, z_{j-1}, 0, z_j, \dots, z_{k+1})}{\prod_{i \in I_{k+1}^j} (L_i^{k+1}|_{z_j=0})^{m_i^{k+1}}},$$

which by assumption (9) lies in $\mathcal{S}_k$. On the other hand,

$$\mathrm{Res}_{z_j=0}^{l}(f(z_1, \dots, z_{k+1})) = \frac{h_l(z_1, \dots, z_{j-1}, 0, z_j, \dots, z_{k+1})}{\prod_{i \in I_{k+1}^j} (L_i^{k+1}|_{z_j=0})^{m_i^{k+1}}}$$

also lies in $\mathcal{S}_k$. The fact that every regularised evaluator at zero also shares the assumption of Proposition 1 then follows from Theorem 1.4, which tells us any evaluator at zero is of the form $\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \sum_{l=1}^{k} \mu_l \, \mathrm{Res}_0^l$. □

Example 1.12. Consider the function $f = \frac{1}{z_1+z_2}$, then $\mathrm{ev}_{z_2=0}^{\mathrm{reg}}(f)(z_1) = \frac{1}{z_1}$ and

$$\mathrm{ev}_{\rightarrow}^{\mathrm{ren}} f = \mathrm{ev}_{z_1=0}^{\mathrm{reg}}(\mathrm{ev}_{z_2=0}^{\mathrm{reg}} f) = 0.$$

Definition 1.13. Given a filtered algebra $\mathcal{S}$ of type (7) which is stable under permutations in the variables, i.e.,

$$\text{if } f \in \mathcal{S}_k \text{ then } f \circ \tau \in \mathcal{S}_k \text{ for all } \tau \in \Sigma_k,$$

we call *symmetric*[7] any generalised evaluator $\Lambda : \mathcal{S} \to \mathbb{C}$ at zero on $\mathcal{S}$ such that

$$\Lambda_k(f \circ \tau) = \Lambda_k(f) \quad \text{for all } \tau \in \Sigma_k, \; f \in \mathcal{S}_k,$$

where Σ_k is the permutation group on k variables.

Example 1.14. Generalised evaluators at zero on a tensor algebra as in (1.9) are clearly symmetric.

[7] Speer only considers symmetric evaluators.

The generalised evaluator $\mathrm{ev}_0^{\mathrm{ren}}$ is not symmetric since

$$\mathrm{ev}_0^{\overset{\mathrm{ren}}{\rightarrow}}\left(\tfrac{z_2}{z_1+z_2}\right) = \mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\tfrac{z_2}{z_1+z_2}\right)\right) = 0$$

whereas

$$\mathrm{ev}_0^{\overset{\mathrm{ren}}{\leftarrow}}\left(\tfrac{z_1}{z_1+z_2}\right) = \mathrm{ev}_{z_1=0}^{\mathrm{reg}}\left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}}\left(\tfrac{z_1}{z_1+z_2}\right)\right) = 1.$$

It can nevertheless be symmetrised. Indeed, any generalised evaluator Λ at zero gives rise to a symmetric generalised evaluator Λ^{sym} on an algebra $\mathcal{S}$ of type (7) which is stable under permutations in the variables, defined on $\mathcal{S}_k$ by

$$\Lambda_k^{\mathrm{sym}} = \tfrac{1}{k!} \sum_{\tau\in\Sigma_k} \Lambda(f\circ\tau).$$

Definition 1.15. A regularised evaluator λ at zero on $\mathcal{S}_1$ gives rise to a generalised evaluator at zero $\lambda^{\mathrm{ren,sym}}$ called the *symmetric generalised evaluator induced by* λ:

$$\lambda^{\mathrm{ren,sym}}|_{\mathcal{S}_k} = \tfrac{1}{k!} \sum_{\tau\in\Sigma_k} \lambda_{z_{\tau(1)}=0}^{\mathrm{reg}} \circ\cdots\circ \lambda_{z_{\tau(k)}=0}^{\mathrm{reg}}.$$

Example 1.16. We compute

$$\mathrm{ev}_0^{\mathrm{ren,sym}}\left(\tfrac{z_1}{z_1+z_2}\right) = \tfrac{1}{2} \neq 1 = \mathrm{ev}_0^{\overset{\mathrm{ren}}{\leftarrow}}\left(\tfrac{z_1}{z_1+z_2}\right).$$

The evaluators we considered above are strongly bound to the filtration corresponding to the choice of coordinates $z_1,\dots,z_k,\dots$. Another choice of coordinates yields another filtration and another generalised evaluator. Let $\vec{u} = A\vec{z}$ for some $(A_k)_{k\in\mathbb{N}}$ in $\mathrm{GL}_\infty(\mathbb{C})$ so that $(u_1,\dots,u_k) = A_k(z_1,\dots,z_k)$ for any positive integer k yields a new set of coordinates. A function $f\in\mathcal{S}_k$ reads

$$f\circ A_k^{-1} = \frac{h\circ A_k^{-1}}{\prod_{i\in I_k}(L_i^k\circ A_k^{-1})^{m_i^k}}.$$

The sets of linear forms $\{L_i^k\circ A_k^{-1},\ i\in I_k\}$ do not a priori form a nested family. Again, we can enlarge each set of level k by multiplying the numerator $h\circ A_k^{-1}$ and the denominator $(\prod_{i\in I_k} L_i^k\circ A_k^{-1})^{m_i^k}$ by appropriately chosen linear forms so as to get a nested family; it can also be further enlarged so as to get a filtered algebra structure for the tensor product.

In this new coordinate system we therefore have a new description of $\mathcal{S}$ as a filtered union $\bigcup_{k=1}^\infty \widetilde{\mathcal{S}}_k$,

$$\widetilde{\mathcal{S}}_k = \{f:\mathbb{C}^k\mapsto\mathbb{C},\ \text{there exists } \widetilde{m}_i^k\in\mathbb{N},\ i\in\tilde{I}_k,\ \text{such that the function } f\textstyle\prod_{i\in\tilde{I}_k}(\widetilde{L}_i^k)^{m_i^k} \text{ is holomorphic around zero}\},$$

so that

$$f \circ A_k^{-1} = \frac{\tilde{h}}{\prod_{i \in I_k} (\widetilde{L}_i^k)^{\tilde{m}_i^k}}.$$

Let $\tilde{\lambda}^{\overset{\text{ren}}{\rightarrow}}$ (resp. $\tilde{\lambda}^{\overset{\text{ren}}{\leftarrow}}$) denote the generalised evaluator induced by λ adapted to this new filtration, i.e., in the new coordinates $u_1, \dots, u_k, \dots$. In general, $\tilde{\lambda}^{\overset{\text{ren}}{\rightarrow}} f$ (resp. $\tilde{\lambda}^{\overset{\text{ren}}{\leftarrow}}$) defines an generalised evaluator at zero different from $\lambda^{\overset{\text{ren}}{\rightarrow}}$ (resp. $\lambda^{\overset{\text{ren}}{\leftarrow}}$).

The fact that there are non-symmetric evaluators such as $\mathrm{ev}_0^{\overset{\text{ren}}{\rightarrow}}$ is a manifestation of the non-invariance of generalised evaluators under permutation matrices.

Example 1.17. Take $f((z_1, z_2)) = \frac{z_1}{z_1+z_2}$ and new coordinates $u_1 = z_1 + z_2$, $u_2 = z_1 - z_2$ in which f reads $\tilde{f}(u_1, u_2) = \frac{u_1+u_2}{2u_1}$. We have $\mathrm{ev}_0^{\overset{\text{ren}}{\rightarrow}} f = 1$ whereas $\widetilde{\mathrm{ev}}_0^{\overset{\text{ren}}{\rightarrow}} f = \mathrm{ev}_0^{\overset{\text{ren}}{\rightarrow}} \tilde{f} = \frac{1}{2}$.

We henceforth focus on generalised evaluators $\lambda^{\overset{\text{ren}}{\rightarrow}}$ induced by a regularised evaluator λ for some specific choice of coordinates.[8]

The following theorem generalises Corollary 1.5 insofar as it expresses a function $f = \frac{h}{(\prod_{i \in I_k} (L_i^k)^{m_i^k})}$ evaluated at zero in terms of jets of the holomorphic function h.

Theorem 1.18. *Let $\mathcal{S}$ be an algebra of type (7) and let $\Lambda := \lambda^{\overset{\text{ren}}{\rightarrow}}$ be a generalised evaluator at zero generated by a regularised evaluator λ at zero on $\mathcal{S}_1$. For any function $f = \frac{h}{(\prod_{i \in I_k} (L_i^k)^{m_i^k})}$ in $\mathcal{S}_k$, where $h\colon \mathbb{C}^k \to \mathbb{C}$ is holomorphic around zero and the m_i^k, $i \in I_k$, correspond to the multiplicity of the poles, the renormalised value $\Lambda(f)$ is a linear expression in the jets of h of order $\sum_{i \in I_k} m_i^k$ at zero.*

In particular, if the poles are simple, starting from an evaluator $\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \mu \operatorname{Res}_0$, a function $f \in \mathcal{S}_k$ evaluated at zero gives rise to a polynomial $\Lambda(f)$ of degree k in μ with coefficient of μ^j given by jets of h at zero of degree no larger than j.

Proof. We show the property by induction on k.

We first observe that given a meromorphic function f in one variable with poles of order m, for any $l \leq m$ we have $\operatorname{Res}_0^l(f) = \mathrm{ev}_0^{\mathrm{reg}}(z^l f(z))$ so that, up to a multiplication of the holomorphic function at the numerator by some monomial z_k^l at step k of the induction procedure, we can assume that the evaluator λ is of the form $\mathrm{ev}_0^{\overset{\text{ren}}{\rightarrow}}$.

The assertion holds for $k = 1$ since $\mathrm{ev}_0^{\mathrm{reg}}(\frac{h(z)}{z^m}) = \frac{h^{(m)}(0)}{m!}$.

Let us assume that the statement holds for $k - 1$. We write

$$\mathrm{ev}_0^{\overset{\text{ren}}{\rightarrow}}(f(z_1, \dots, z_k)) = (\mathrm{ev}_{z_1=0}^{\mathrm{reg}} \cdots \mathrm{ev}_{z_{k-1}=0}^{\mathrm{reg}}(\mathrm{ev}_{z_k=0}^{\mathrm{reg}} f(z_1, \dots, z_k)))$$

[8] One might raise the question as what other generalised evaluators exist. To my knowledge, the classification of generalised evaluators at zero is an open question.

and apply the induction assumption to $\mathrm{ev}^{\mathrm{reg}}_{z_k=0} f(z_1, \ldots, z_k)$, which now depends on $k-1$ variables.

If the linear forms $L_i^k(z_1, \ldots, z_k) = \sum_{i=1}^{k} a_{k,i} z_i$ are defined by rational coefficients $a_{k,i}$, then the evaluator yields a rational linear expression in the jets of h at zero. □

2 Divergent integrals and sums of homogeneous functions

Given a complex number a with real part smaller than -1 the integral $\int_1^\infty x^a\, dx$ and the sum $\sum_{n=1}^\infty n^a$ converge. Our aim is to make sense of this integral and this sum when the real part is larger than or equal to -1 and to present different ways of regularising this integral. In spite of the seeming simplicity of the problem considered here, namely how to regularise one-dimensional integrals of homogeneous functions, this toy model reflects many of the features encountered when regularising integrals of symbols in higher dimensions. In particular, we shall see how discrepancies arise that are elementary occurrences of more general "anomalous" phenomena. There are at least three ways to extract finite parts from homogeneous distributions inspired by mathematical procedures on the one hand:

(1) Hadamard's finite part method (see e.g. [35], Example 2, chapter II) which amounts to approximating the domain of integration $[1, \infty[$ by bounded intervals $[1, R]$ with R tending to infinity.

(2) Riesz' meromorphic approach (see e.g. [35], Example 3, chapter II) which amounts here to approximating the homogeneous function x^a in the integrand by the function x^{a-z} with z tending to zero.

(3) Approximating the homogeneous function x^a in the integrand by the function $x^a\, e^{-\varepsilon x}$ with ε tending to zero.

These have their counterparts in physics, respectively called cut-off, dimensional (here the dimension is 1) and heat-kernel regularisation.

2.1 Cut-off regularisation. The idea is to extract a finite part when R tends to ∞ of $\int_1^R x^a\, dx$. We distinguish two cases:

(1) For $a = -1$ we have

$$\int_1^R \frac{1}{x}\, dx = [\log x]_1^R = \log R,$$

an expression which diverges as $R \to \infty$ and from which we extract a finite part defined by

$$⨍_1^\infty \frac{1}{x}\, dx := \mathrm{fp}_{R\to\infty} \int_1^R \frac{1}{x}\, dx := 0.$$

(2) When $a \neq -1$

$$\int_1^R x^a\,dx = \left[\frac{x^{a+1}}{a+1}\right]_1^R = \frac{R^{a+1}}{a+1} - \frac{1}{a+1},$$

an expression which also diverges as $R \to \infty$ and from which we extract a finite part defined by

$$-\!\!\!\!\!\!\int_1^\infty x^a dx := \mathrm{fp}_{R\to\infty} \int_1^R x^a dx := -\frac{1}{a+1}.$$

This leads to the following elementary result.

Lemma 2.1. *The integral*

$$\int_1^R x^a\,dx = (1-\delta_{a+1})\frac{R^{a+1}}{a+1} - \frac{1-\delta_{a+1}}{a+1} + \delta_{a+1}\log R$$

diverges as $R \to \infty$ when $\mathrm{Re}(a) \geq -1$. The ***cut-off regularised integral*** *given by its finite part when $R \to \infty$,*

$$-\!\!\!\!\!\!\int_1^\infty x^a dx := \mathrm{fp}_{R\to\infty} \int_1^R x^a dx := \frac{\delta_{a+1}-1}{a+1},$$

vanishes when $a = -1$ and coincides with $\int_1^\infty x^a dx$ when $\mathrm{Re}(a) < -1$.

A ***rescaling*** *$R \to \lambda R$ for some $\lambda > 0$ only modifies the finite part if $a = -1$:*

$$\mathrm{fp}_{R\to\infty} \int_1^{\lambda R} x^a dx = -\!\!\!\!\!\!\int_1^\infty x^a dx + \delta_{a+1}\log\lambda. \tag{10}$$

Remark 2.2. More generally, an affine transformation $R \to \lambda R + \alpha$ changes the finite part by adding to it $\frac{1-\delta_{a+1}}{a+1}\alpha^{a+1} + \delta_{a+1}\log\lambda$, so that if $\lambda = 1$ the finite part only changes for $a \neq -1$.

2.2 Riesz regularisation. The idea is to extract a finite part when $z \to 0$ of a meromorphic extension of the map $z \mapsto \int_1^\infty x^{a-z} dx$, which is holomorphic on the half-plane $\mathrm{Re}(a) < -1$. The cut-off integral $-\!\!\!\!\!\int_1^\infty x^{a-z} dx$ provides a meromorphic extension and we define the finite part as the constant term of its Laurent expansion.

Proposition 2.3. (1) *The cut-off regularised integral*

$$-\!\!\!\!\!\!\int_1^\infty x^{a-z} dx = \frac{\delta_{a-z+1}-1}{a-z+1}$$

defines a meromorphic map with simple poles in $[-1, +\infty[\, \cap \mathbb{Z}$ and the residue at $z = 0$ reads

$$\mathrm{Res}_0 -\!\!\!\!\!\!\int_1^\infty x^{a-z} dx = \delta_{a+1},$$

so that the map $z \mapsto \fint x^{a-z}\,dx$ *is holomorphic at* 0 *whenever* $a \neq -1$*. It is holomorphic on the half-plane* $\mathrm{Re}(z) > \mathrm{Re}(a) + 1$*, on which it coincides with the ordinary integral* $\int_1^\infty x^{a-z}\,dx$.

(2) *For any* $a \in \mathbb{C}$ *the* ***Riesz regularised integral****, defined as the finite part at* $z = 0$*, which reads*

$$\fint_1^{\infty,\mathrm{Riesz}} x^a\,dx := \mathrm{ev}_0^{\mathrm{reg}} \fint_1^\infty x^{a-z}\,dx = \frac{\delta_{a+1}-1}{a+1},$$

coincides with the cut-off regularised integral

$$\fint_1^{\infty,\mathrm{Riesz}} x^a\,dx = \fint_1^\infty x^a\,dx,$$

so that we henceforth drop the superscript Riesz.

(3) *A holomorphic reparametrisation* $z \mapsto f(z)$ *with* $f(0) = 0$ *and* $f'(0) \neq 0$[9] *only modifies this finite part when* $a = -1$:

$$\mathrm{ev}_0^{\mathrm{reg}} \fint_1^\infty x^{a-f(z)}dx = \fint_1^\infty x^a\,dx + \delta_{a+1}\mu = \fint_1^\infty x^a\,dx + \mu\,\mathrm{Res}_0\left(\fint_1^\infty x^{a-z}\,dx\right), \tag{11}$$

where we have set $\mu = -\frac{f''(0)}{2(f'(0))^2}$.

(4) *Let* H *be a holomorphic function such that* $H(0) = 1$*. Then the map* $z \mapsto H(z) \fint x^{a-z}\,dx$ *is meromorphic with a simple pole at* $z = 0$ *given by* δ_{a+1} *and coincides with the holomorphic map* $z \mapsto H(z) \fint x^{a-z}\,dx$ *on the half-plane* $\mathrm{Re}(z) > \mathrm{Re}(a) + 1$*. Its finite part at* $z = 0$ *reads*

$$\begin{aligned}\mathrm{ev}_0^{\mathrm{reg}}\Big(H(z) \fint_1^\infty x^{a-z}\,dx\Big) &= \fint_1^\infty x^a\,dx + \delta_{a+1} H'(0)\\ &= \fint_1^\infty x^a\,dx + H'(0)\,\mathrm{Res}_0 \fint_1^\infty x^{a-z}\,dx.\end{aligned}$$

In particular, if $a \neq -1$ *then the regularised value is independent of* H:

$$\mathrm{ev}_0^{\mathrm{reg}}\left(H(z) \fint_1^\infty x^{a-z}\,dx\right) = \fint_1^\infty x^a\,dx.$$

Remark 2.4. Note the analogy between (11) and (10) when setting $\mu = \log\lambda$.

Proof. The proof follows in a straightforward manner from Lemma 2.1. The third item follows from $\frac{1}{f(z)} = \frac{1}{f'(0)z}(1 - \frac{f''(0}{2f'(0)}z + o(z))$. □

[9] E.g., $f(z) = \frac{z}{z+\mu z}$.

2.3 Heat-kernel regularisation. We now consider a third approach to regularise $\int_1^\infty x^a\,dx$, a method inspired by heat-kernel approximation familiar both to mathematicians and physicists. We compare this method with the two approaches described previously.

The idea is to extract a finite part of $\int_1^\infty x^a e^{-\varepsilon x^p}\,dx$ for some positive integer p as $\varepsilon \to 0$. We therefore need to control the asymptotics[10] of the map $\varepsilon \mapsto \int_1^\infty x^a e^{-\varepsilon x^p}\,dx$. Let us introduce some notation. $\mathcal{A}[\varepsilon]$ denotes the algebra of analytic functions in the variable ε and for any complex number α and any positive integer l we set

$$\mathcal{A}[\varepsilon]\varepsilon^\alpha \log^l \varepsilon := \{f(\varepsilon)\varepsilon^a \log^l \varepsilon,\ f \in \mathcal{A}[\varepsilon]\}.$$

Proposition 2.5. (1) *If* $\frac{a+1}{p} \notin -\mathbb{N} \cup \{0\}$, *then* $\int_1^\infty x^a e^{-\varepsilon x^p}\,dx$ *lies in*

$$\mathcal{A}[\varepsilon]\varepsilon^{-\frac{a+1}{p}} \oplus \mathcal{A}[\varepsilon]$$

with finite part independent of p *given by*

$$\mathrm{fp}_{\varepsilon\to 0} \int_1^\infty x^a e^{-\varepsilon x^p}\,dx = -\frac{1}{a+1}.$$

(2) *If* $a = -1$ *then* $\int_1^\infty x^{-1} e^{-\varepsilon x^p}\,dx$ *lies in* $\bigoplus_{l=0}^1 \mathcal{A}[\varepsilon]\log^l \varepsilon$ *and the finite part is given by*

$$\mathrm{fp}_{\varepsilon\to 0} \int_1^\infty x^{-1} e^{-\varepsilon x^p}\,dx = -\frac{\gamma}{p},$$

where

$$\gamma := -\int_0^\infty (\log x) e^{-x}\,dx$$

is the Euler constant.

(3) *If* $\frac{a+1}{p} = -k$ *for some integer* $k > 0$, *then* $\int_1^\infty x^{-k} e^{-\varepsilon x^p}\,dx$ *lies in*

$$\mathcal{A}[\varepsilon]\varepsilon^{k-1} \log \varepsilon \oplus \mathcal{A}[\varepsilon]$$

with finite part given by

$$\mathrm{fp}_{\varepsilon\to 0} \int_1^\infty x^a e^{-\varepsilon x^p}\,dx = \frac{1}{p(k-1)}.$$

Proof. We first study the case $p = 1$ from which the case $p > 1$ will easily follow.

(1) The case $p = 1$. We further distinguish three subcases, namely $a \notin -\mathbb{N}$, $a = -1$ and $a = -k$, $k \in \mathbb{N} - \{1\}$.

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[10]Let me address my thanks to Bin Zhang for helping me clarify technical aspects of the following computations.

- If $a \notin -\mathbb{N}$, there is some positive integer k such that $a + k > 1$ and $a + 1 \neq 0$, $a + 2 \neq 0$, ..., $a + k - 1 \neq 0$. Repeated integration by parts yields:

$$\begin{aligned}\int_1^\infty x^a e^{-\varepsilon x}\,dx &= \varepsilon^{-(a+1)} \int_\varepsilon^\infty x^a e^{-x}\,dx \\ &= \frac{\varepsilon^{-(a+1)}}{(a+1)\dots(a+k)} \int_\varepsilon^\infty x^{a+k} e^{-x}\,dx - \frac{\varepsilon^{k-1}}{(a+1)\dots(a+k)} e^{-\varepsilon} \\ &\quad \dots - \frac{\varepsilon^{j-1}}{(a+1)\dots(a+j)} e^{-\varepsilon} - \dots - \frac{e^{-\varepsilon}}{a+1} \\ &= \frac{\varepsilon^{-(a+1)}}{(a+1)\dots(a+k)} \left[\int_0^\infty x^{a+k} e^{-x}\,dx - \int_0^\varepsilon x^{a+k} e^{-x}\,dx\right] \\ &\quad - \sum_{j=1}^k \frac{e^{-\varepsilon}\varepsilon^{j-1}}{(a+1)\dots(a+j)} \\ &= \frac{1}{(a+1)\dots(a+k)} \left[\varepsilon^{-(a+1)} \int_0^\infty x^{a+k} e^{-x}\,dx \right. \\ &\quad \left. - \varepsilon^k \int_0^1 x^{a+k} e^{-\varepsilon x}\,dx\right] - \sum_{j=1}^k \frac{e^{-\varepsilon}\varepsilon^{j-1}}{(a+1)\dots(a+j)}.\end{aligned}$$

Since the integral $\int_0^\infty x^{a+k} e^{-x}\,dx$ converges, the first term lies in $\mathcal{A}[\varepsilon]\varepsilon^{-(a+1)}$ and since $a \neq -1$ this first term does not contribute to the finite part. Since the map $\varepsilon \mapsto \int_0^1 x^{a+k} e^{-\varepsilon x}\,dx$ is analytic, the second term lies in $\mathcal{A}[\varepsilon]$ and since $k > 0$ neither does this second term contribute to the finite part. Only the last term, which lies in $\mathcal{A}[\varepsilon]$, contributes to the finite part by $-\frac{1}{a+1}$. Hence the map $\varepsilon \mapsto \int_1^\infty x^a e^{-\varepsilon x^p}\,dx$ lies in $\mathcal{A}[\varepsilon]\varepsilon^{-(a+1)} \oplus \mathcal{A}[\varepsilon]$ and the finite part reads

$$\mathrm{fp}_{\varepsilon\to 0} \int_1^\infty x^a e^{-\varepsilon x}\,dx = -\frac{1}{a+1}.$$

- If $a = -1$ then by integration by parts

$$\int_1^\infty x^{-1} e^{-\varepsilon x}\,dx = \int_\varepsilon^\infty x^{-1} e^{-x}\,dx = \int_\varepsilon^\infty (\log x) e^{-x}\,dx + \log\varepsilon$$

lies in $\bigoplus_{l=0}^1 \mathcal{A}[\varepsilon] \log^l \varepsilon$ and the finite part is given by

$$\mathrm{fp}_{\varepsilon\to 0} \int_1^\infty x^{-1} e^{-\varepsilon x}\,dx = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty (\log x) e^{-x}\,dx = -\gamma.$$

• If $a=-k$, for some integer $k>1$, then by iterated integrations by parts

$$\begin{aligned}\int_1^\infty x^{-k}e^{-\varepsilon x}dx &= \varepsilon^{k-1}\int_\varepsilon^\infty x^{-k}e^{-x}dx\\ &= \frac{(-1)^{k+1}\varepsilon^{k-1}}{(k-1)!}\int_\varepsilon^\infty(\log x)\,e^{-x}dx+(-1)^{k+1}\frac{\log\varepsilon\, e^{-\varepsilon}\varepsilon^{k-1}}{(k-1)!}\\ &\quad+\frac{1}{k-1}e^{-\varepsilon}+\cdots+\frac{(-1)^{j-1}\varepsilon^{j-1}}{(k-1)\ldots(k-j)}e^{-\varepsilon}+\cdots\\ &\quad\cdots+\frac{(-1)^{k-2}\varepsilon^{k-2}}{(k-1)!}e^{-\varepsilon},\end{aligned}$$

which lies in $\mathcal{A}[\varepsilon]\varepsilon^{k-1}\log\varepsilon\oplus\mathcal{A}[\varepsilon]$. The finite part is given by

$$\mathrm{fp}_{\varepsilon\to 0}\int_1^\infty x^{-k}e^{-\varepsilon x}dx=\frac{1}{k-1}.$$

(2) The case $p>1$. A change of variable $y:=x^p$ yields

$$\int_1^\infty x^a e^{-\varepsilon x^p}\,dx=\frac{1}{p}\int_1^\infty x^{\frac{a+1}{p}-1}e^{-\varepsilon x}\,dx.$$

Replacing a in the asymptotics by $\frac{a+1}{p}-1$ and dividing the finite parts by p then yields the result. □

Definition 2.6. Given $a\in\mathbb{C}$, the *heat-kernel regularised integral* is defined by

$$\int_1^{\infty,\mathrm{HK}}x^a dx:=\mathrm{fp}_{\varepsilon\to 0}\int_1^\infty x^a e^{-\varepsilon x}dx=\mathrm{fp}_{\varepsilon\to 0}\left(\varepsilon^{-(a+1)}\int_\varepsilon^\infty x^a e^{-x}dx\right)$$

so that, by the above proposition,

$$\int_1^{\infty,\mathrm{HK}}x^a dx=-\frac{1-\delta_{a+1}}{a+1}-\gamma\delta_{a+1}.$$

Combining the various results of this section leads to the following statement.

Theorem 2.7. *The cut-off, Riesz and heat-kernel type regularised values of integrals of homogeneous functions* $\int_1^\infty x^a dx$ *coincide for all* $a\neq -1$ *but do not all coincide* *for* $a = -1$:

$$\begin{aligned}
⨍_1^\infty x^a\,dx &= \frac{\delta_{a+1} - 1}{a+1} \\
&= \int_1^{\infty,\mathrm{HK}} x^a\,dx + \delta_{a+1}\gamma \\
&= \int_1^{\infty,\mathrm{HK}} x^a\,dx + \gamma\,\mathrm{Res}_0\left(⨍_1^\infty x^{a-z}\,dx\right) \\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(H(z)⨍_1^\infty x^{a-z}dx\right) - H'(0)\,\mathrm{Res}_0\left(⨍_1^\infty x^{a-z}\,dx\right) \\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(H(z)⨍_1^\infty x^{a-z}dx\right) - H'(0)\,\delta_{a+1},
\end{aligned}$$

where H *is any holomorphic function such that* $H(0) = 1$.

Remark 2.8. For future use, it is useful to notice that all the regularised integrals, whether by cut-off, Riesz or heat-kernel procedures, coincide for homogeneous symbols of non-integer order.

2.4 The Euler–Maclaurin formula. The classical Euler–Maclaurin formula, which relates a sum to an integral, involves the Bernoulli numbers defined by the following Taylor expansion at $t = 0$ (see e.g [7]):

$$\frac{t}{e^t - 1} := \sum_{n=0}^\infty B_n \frac{t^n}{n!}.$$

Since $\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2}\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}$ is an even function, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for any positive integer k.

Remark 2.9. In view of a generalisation to higher dimensions, it is useful to observe that $\frac{t}{e^t-1} = \mathrm{Td}(-t)$, where $\mathrm{Td}(t) := \frac{t}{1-e^{-t}}$ is the *Todd function*, so that

$$\mathrm{Td}(t) = \sum_{n=0}^\infty (-1)^n\,B_n\,\frac{t^n}{n!} = \frac{t}{2} + \sum_{k=0}^\infty B_{2k}\,\frac{t^{2k}}{(2k)!} = 1 + \frac{t}{2} + \sum_{k=2}^\infty (-1)^k\,\frac{B_k}{k!}.$$

One computes the following values of the Bernoulli numbers (see e.g. [7]): $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$.

Bernoulli polynomials are defined similarly by

$$\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!} = \frac{t\,e^{tx}}{e^t - 1} \tag{12}$$

so that in particular $B_n(0) = B_n$.

This initial condition combined with the differential equations obtained from differentiating (12) with respect to x

$$\partial_x B_n(x) = n B_{n-1}(x),$$

completely determine the Bernoulli polynomials and that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

Note that $B_1(x) = -\frac{1}{2} + x$.

Furthermore, $\sum_{n=0}^{\infty} (B_n(1) - B_n(0)) \frac{t^n}{n!} = \frac{te^t - t}{e^t - 1} = t$ so that

$$B_1(1) = B_1(0) + 1 \quad \text{and} \quad B_n(1) = B_n(0) \quad \text{for all } n \geq 2.$$

Since $B_n(1) = B_n$ for any $n \geq 2$, setting $x = 1$ we have

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad \text{for all } n \geq 2.$$

Let us recall the Euler–Maclaurin formula (see e.g. [19]).

Proposition 2.10. *For any function f in $C^\infty(\mathbb{R})$ and any two integers $M < N$,*

$$\begin{aligned}\sum_{n=M}^{N} f(n) = \frac{f(M) + f(N)}{2} &+ \int_M^N f(x)\,dx \\ &+ \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} (f^{(k-1)}(N) - f^{(k-1)}(M)) \\ &+ \frac{(-1)^{K-1}}{K!} \int_M^N \overline{B_K}(x)\, f^{(K)}(x)\,dx\end{aligned}$$

with $\overline{B_k}(x) = B_k(x - [x])$ and where K is any positive integer larger than 1.

Proof. Set, for convenience,

$$S_K(f) := \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} (f^{(k-1)}(N) - f^{(k-1)}(M)),$$
$$R_K(f) := \frac{(-1)^{K-1}}{K!} \int_M^N \overline{B_K}(x)\, f^{(K)}(x)\,dx$$

and observe that

$$S_K(f) + R_K(f) = S_{K+1}(f) + R_{K+1}(f).$$

The proof then follows by induction. □

Remark 2.11. The index K in the Euler–Maclaurin formula can be chosen arbitrarily large.

Example 2.12. If f is polynomial, then the remainder term $R_K(f)$ vanishes for large enough K and we have

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)\,dx + \left[(\mathrm{Td}(\partial_{h_1})\,\mathrm{Td}(\partial_{h_2}) - \mathrm{Id}) \int_{M-h_1}^{N+h_2} f(\xi)\,d\xi\right]_{|h_1=h_2=0}.$$

Example 2.13. For a homogeneous function $f(x) = x^a$ we have

$$\begin{aligned}\sum_{n=M}^{N} n^a = \frac{M^a + N^a}{2} &+ \int_1^N x^a dx \\ &+ \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!} a(a-1)\dots(a-k+2)(N^{a-k+1} - M^{a-k+1}) \\ &+ \frac{(-1)^{K-1}}{K!}\int_M^N \overline{B_K}(x)(a-1)\dots(a-K+2)x^{a-K+1}dx.\end{aligned} \tag{13}$$

2.5 Cut-off and Riesz regularised sums of homogeneous functions. Regularised discrete sums can be defined in terms of regularised integrals using the Euler–Maclaurin formula (see e.g. [27]).

Given a complex number a, let us choose the integer K large enough in (13) for $\int_1^\infty \overline{B_K}(x)x^{a-K+1}dx$ to converge. Further setting $M = 1$ yields an asymptotic expansion as N tends to infinity from which we can extract a finite part of the homogeneous function x^a over the positive integers and defined in terms of the cut-off regularised integral $⨍_1^\infty x^a dx$ as follows.

Definition 2.14. The *cut-off regularised sum* of the homogeneous function x^a over the positive integers is defined by

$$\begin{aligned}⨊_{n=1}^{\infty} n^a := \frac{1+\delta_a}{2} + ⨍_1^\infty x^a dx + \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!} a(a-1)\dots(a-k+2)(\delta_{a-k+1} - 1) \\ + a(a-1)\dots(a-K+2)\frac{(-1)^{K-1}}{K!}\int_1^\infty \overline{B_K}(x)x^{a-K+1}dx,\end{aligned}$$

independently of the positive integer K provided it is chosen large enough.

Replacing a in the exponent by $a - z$ yields a holomorphic map on the half-plane $\mathrm{Re}(z) < \mathrm{Re}(a) + 1$ with a meromorphic extension to the whole complex plane given by

$$\sum_{n=1}^{\infty} n^{a-z}$$
$$:= \frac{1}{2} + \int\!\!\!\!-_1^{\infty} x^{a-z}\,dx - \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!}(a-z)(a-z-1)\dots(a-z-k+2)$$
$$+ \frac{(-1)^{K-1}}{K!}\int_1^{\infty} \overline{B_K}(x)(a-z)(a-z-1)\dots(a-z-K+2)x^{a-z-K+1}\,dx.$$

It has simple poles in $\{a+1-k, k \in \mathbb{Z}_+\}$ given by the poles of the meromorphic map $-\!\!\!\!\int_1^\infty x^{a-z}\,dx$. Its residue at $z=0$ is given by

$$\mathrm{Res}_0\Big(\sum_{n=1}^{\infty} n^{a-z}\Big) = \delta_{a+1}.$$

The finite part at $z=0$ gives rise to the following definition.

Definition 2.15. The *Riesz regularised sum* of the homogeneous function x^a over the positive integers is defined by

$$\sum_{n=1}^{\infty,\mathrm{Riesz}} n^a := \frac{1}{2} + \int\!\!\!\!-_1^{\infty,\mathrm{Riesz}} x^a\,dx - \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!}a(a-1)\dots(a-k+2)$$
$$+ a(a-1)\dots(a-K+2)\frac{(-1)^{K-1}}{K!}\int_1^{\infty} \overline{B_K}(x)x^{a-K+1}\,dx$$
$$= \frac{1}{2} + \int\!\!\!\!-_1^{\infty} x^a\,dx - \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!}a(a-1)\dots(a-k+2)$$
$$+ a(a-1)\dots(a-K+2)\frac{(-1)^{K-1}}{K!}\int_1^{\infty} \overline{B_K}(x)\,x^{a-K+1}\,dx,$$

independently of the positive integer K provided it is chosen large enough.

Remark 2.16. Here we used the fact that the cut-off and Riesz regularised integrals coincide. By contrast, cut-off and Riesz regularised discrete sums do not coincide for non-negative integers a but they do for non-integer a.

We now present an alternative method to build a meromorphic extension of the holomorphic map $z \mapsto \sum_{n=1}^{\infty} n^{a-z}$ on the half plane $\mathrm{Re}(z) > \mathrm{Re}(a)+1$ by means of a Mellin transform

$$f \mapsto \mathcal{M}(f)(s) := \frac{1}{\Gamma(s)}\int_0^{\infty} \varepsilon^{s-1} f(\lambda)\,d\varepsilon$$

defined for any smooth functions f on the non-negative real line with exponential decay at infinity. This method will be generalised to multiple sums on cones (see Proposition 5.4). Observing that

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \varepsilon^{s-1} e^{-\varepsilon\lambda} d\varepsilon,$$

for $\mathrm{Re}(s) > 1$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-s} &= \frac{1}{\Gamma(s)} \int_0^\infty \varepsilon^{s-1} \sum_{n=1}^{\infty} e^{-\varepsilon n} d\varepsilon \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \varepsilon^{s-1} e^{-\varepsilon} \frac{1}{1-e^{-\varepsilon}} d\varepsilon \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \varepsilon^{s-2} eps^{-\varepsilon} \frac{\varepsilon}{1-e^{-\varepsilon}} d\varepsilon \\ &= \frac{\mathcal{M}(h)(s-1)}{s-1}, \end{aligned} \tag{14}$$

where we have set $h(\varepsilon) := \frac{\varepsilon\, e^{-\varepsilon}}{1-e^{-\varepsilon}}$. Since h is smooth and exponentially decreasing at infinity and since $h(\varepsilon) = \sum_{k=1}^{K} a_k\, \varepsilon^k + o(\varepsilon^K)$ for any non-negative integer K, its Mellin transform $\mathcal{M}(h)(s-1)$ is well defined and gives rise to a meromorphic map on $\mathbb{C}$ with simple poles in the set $-\mathbb{N}_0$ (see e.g. Lemma 9.35 in [3]). Hence $s \mapsto \sum_{n=1}^{\infty} n^{-s} := \frac{\mathcal{M}(h)(s-1)}{s-1}$ defines a meromorphic extension of $s \mapsto \sum_{n=1}^{\infty} n^{-s}$ to the complex plane with simple poles in the set $1 - \mathbb{N}_0$.

2.6 Heat-kernel regularised sums of homogeneous functions. Given a positive real number ε, a positive integer p and a complex number a, the function $f_\varepsilon^{a,p}(x) = x^a\, e^{-\varepsilon\, x^p}$ defines a Schwartz function on $\mathbb{R}_+$. Setting $M = 1$ and letting N tend to infinity in the Euler–Maclaurin formula yields

$$\begin{aligned} \sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} = \frac{e^{-\varepsilon}}{2} + \int_1^\infty x^a e^{-\varepsilon x^p} dx + \sum_{k=2}^{K} (-1)^{k+1} \frac{B_k}{k!} \partial_x^{k-1} f_\varepsilon^{a,p}(1) \\ + \frac{(-1)^{K-1}}{K!} \int_1^\infty \overline{B_K}(x) \partial_x^K f_\varepsilon^{a,p}(x)\, dx \end{aligned} \tag{15}$$

since the derivatives $\partial_x^k f_\varepsilon^{a,p}(N)$ converge exponentially fast to zero as $x \to \infty$. In particular, this implies that the integral $\int_1^N \overline{B_K}(x) \partial_K f_\varepsilon^{a,p}(x) dx$ arising in the Euler–Maclaurin formula converges to $\int_1^\infty \overline{B_K}(x) \partial_K f_\varepsilon^{a,p}(x) dx$ as $N \to \infty$.

We now define the heat-kernel regularised discrete sum of the homogeneous function in a similar manner to the heat-kernel regularised integral.

Proposition 2.17. *Given a complex number a, the expression $\sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} dx - \int_1^{\infty} x^a e^{-\varepsilon x^p} dx$ converges as $\varepsilon \to 0$ and the following holds:*

(1) *For $a \notin \mathbb{N}$ and K a large enough integer,*

$$\begin{aligned}\mathrm{fp}_{\varepsilon=0} \sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} &= \frac{1}{2} + \mathrm{fp}_{\varepsilon=0} \int_1^{\infty} x^a e^{-\varepsilon x^p} dx \\ &\quad + \sum_{k=2}^{K} (-1)^{k+1} B_k \frac{a(a-1)\dots(a-k+2)}{k!} \\ &\quad + (-1)^{K-1} \frac{a(a-1)\dots(a-k+1)}{K!} \int_1^{\infty} \overline{B_K}(x)\, x^{a-K} dx,\end{aligned}$$

and for integer values $K \in \mathbb{N} \cup \{0\}$,

$$\mathrm{fp}_{\varepsilon=0} \sum_{n=1}^{\infty} n^K e^{-\varepsilon n^p} = \frac{1}{2} + \mathrm{fp}_{\varepsilon=0} \int_1^{\infty} x^{K+2} e^{-\varepsilon x^p} dx + \sum_{k=2}^{K+1} (-1)^{k+1} C_{K+2}^{k-1} \frac{B_k}{k}.$$

(2) *When $p = 1$ and $a \notin \mathbb{N}$, the* ***heat-kernel regularised sum*** *reads*

$$\begin{aligned}\sum_{n=1}^{\infty,\mathrm{HK}} n^a &:= \mathrm{fp}_{\varepsilon=0} \sum_{n=1}^{\infty} n^a e^{-\varepsilon n} \\ &= \frac{1}{2} + \frac{1-\delta_{a+1}}{a+1} + \sum_{k=2}^{K} (-1)^{k+1} B_k \frac{a(a-1)\dots(a-k+2)}{k!} \\ &\quad + (-1)^{K-1} \frac{a(a-1)\dots(a-k+1)}{K!} \int_1^{\infty} \overline{B_K}(x)\, x^{a-K} dx,\end{aligned}$$

and for $a = K \in \mathbb{N} \cup \{0\}$,

$$\sum_{n=1}^{\infty,\mathrm{HK}} n^a := \mathrm{fp}_{\varepsilon=0} \sum_{n=1}^{\infty} n^a e^{-\varepsilon n} = \frac{K-1}{2(K+1)} + \sum_{k=2}^{K+1} (-1)^{k+1} C_{K+2}^{k-1} \frac{B_k}{k}.$$

Proof. (1) We need to analyse the various terms arising in (15) and hence the express the higher derivatives of $f_\varepsilon^{a,p}$. Since $f_\varepsilon^{a,p} = f_\varepsilon^{\frac{a}{p},1} \circ g_p$ with $g_p(x) = x^p$,

$$\begin{aligned}\partial_x^k f_\varepsilon^{a,p} &= \partial_x^{k-1}\big(\big(\partial_x f_\varepsilon^{\frac{a}{p},1} \circ g_p\big) g_p'\big) \\ &= \partial_x^{k-1}\big(\big(\big(\tfrac{a}{p} f_\varepsilon^{\frac{a-p}{p},1} - \varepsilon f_\varepsilon^{\frac{a}{p},1}\big) \circ g_p\big) g_p'\big) \\ &= \partial_x^{k-1}\big(\big(\tfrac{a}{p} f_\varepsilon^{a-p,p} - \varepsilon f_\varepsilon^{a,p}\big) g_p'\big) \\ &= \sum_{i=0}^{k-1} C_{k-1}^i \big(\tfrac{a}{p} \partial_x^i f_\varepsilon^{a-p,p} - \varepsilon \partial_x^i f_\varepsilon^{a,p}\big) \partial_x^{k-i} g_p.\end{aligned} \tag{16}$$

We claim that $\partial_x^k f_\varepsilon^{a,p}$ is a finite linear combination of expressions of the type $\varepsilon^j x^{jp+a-k} e^{-\varepsilon x^p}$, with j in $\{0,\dots,k\}$ and that $\partial_x^k f_\varepsilon^{a,p}(x)_{|\varepsilon=0} = a(a-1)\dots(a-k+1)x^{a-k}$ comes from the $j=0$ term. Note that when $a=J$ is an integer, then this term vanishes for $k \geq J+1$ since the corresponding derivatives vanish.

The property holds for $k=1$ since $\partial_x f_\varepsilon^{a,p}(x) = a x^{a-1} e^{-\varepsilon x^p} - p\varepsilon x^{a+p-1} e^{-\varepsilon x^p}$ and $\partial_x f_\varepsilon^{a,p}(x)_{|\varepsilon=0} = a x^{a-1}$. Now, assuming that these properties hold for $k-1$, by (16), one easily checks that they hold for k.

In particular,

$$\Big(\sum_{k=2}^{K}(-1)^{k+1}\tfrac{B_k}{k!}\partial_x^{k-1} f_\varepsilon^{a,p}(1)\Big)_{|\varepsilon=0} = \sum_{k=2}^{K}(-1)^{k+1}\tfrac{B_k}{k!}a(a-1)\dots(a-k+2).$$

Furthermore, the integrands $\overline{B_K}(x)\partial_x^K f_\varepsilon^{a,p}(1)(x)$ are linear combinations of expressions

$$\varepsilon^j\,\overline{B_K}(x)\,x^{jp+a-K}\,e^{-\varepsilon x^p}, \qquad j \in \{0,\dots,K\},$$

and $|\overline{B_K}(x)\,x^{jp+a-K}e^{-\varepsilon x^p}| \leq C|x^{jp+a-K}e^{-\varepsilon x^p}|$ for some constant C.

For large enough K the expression $\varepsilon^j\int_1^\infty \overline{B_K}(x)\,x^{jp+a-K}e^{-\varepsilon x^p}\,dx$ converges as $\varepsilon \to 0$ and $\int_1^\infty \overline{B_K}(x)\partial_x^K f_\varepsilon^{a,p}(1)(x)$ therefore converges. Its limit corresponds to the $j=0$ term

$$a(a-1)\dots(a-K+1)\int_1^\infty \overline{B_K}(x)\,x^{a-K}e^{-\varepsilon x^p}$$

since all other terms vanish in the limit. This term vanishes if a is an integer smaller than K.

It follows from the above discussion that the difference

$$\sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} - \int_1^\infty x^a e^{-\varepsilon x^p}\,dx$$

converges as $\varepsilon \to 0$. Taking finite parts on either side of (15) yields for $a \notin \mathbb{N}$ and K chosen large enough

$$\begin{aligned}\mathrm{fp}_{\varepsilon=0}\sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} &= \frac{1}{2} + \mathrm{fp}_{\varepsilon=0}\int_1^\infty x^a e^{-\varepsilon x^p}\,dx \\ &\quad + \sum_{k=2}^{K}(-1)^{k+1}B_k\,\frac{a(a-1)\dots(a-k+2)}{k!} \\ &\quad + (-1)^{K-1}\frac{a(a-1)\dots(a-k+1)}{K!}\int_1^\infty \overline{B_K}(x)\,x^{a-K}\,dx.\end{aligned}$$

For integer values $a = J \in \mathbb{N}$, we take $K = J+2$ as before to ensure convergence of the remainder term. This leads to

$$\mathrm{fp}_{\varepsilon=0}\sum_{n=1}^{\infty} n^a e^{-\varepsilon n^p} = \frac{1}{2} + \mathrm{fp}_{\varepsilon=0}\int_1^\infty x^{J+2}\,e^{-\varepsilon x^p}\,dx + \sum_{k=2}^{J+1}(-1)^{k+1}C_{J+2}^{k-1}\frac{B_k}{k}.$$

This yields the first part of the proposition.

(2) Combining these results with the results on heat-kernel regularised integrals

$$\int_1^{\infty,\mathrm{HK}} x^a = \mathrm{fp}_{\varepsilon\to 0}\int_1^{\infty} x^a e^{-\varepsilon x}\,dx = \frac{\delta_{a+1}-1}{a+1} - \gamma\delta_{a+1}$$

yields the second part of the proposition. □

Combining the results of this subsection with the previous results on regularised integrals leads to the following conclusion which compares with Theorem 2.7.

Theorem 2.18. *Cut-off, Riesz and heat-kernel type regularised values of sums of homogeneous functions $\sum_{n=1}^{\infty} n^a\, dx$ coincide for all $a \notin \mathbb{N}\cup\{-1,0\}$ since for all $a \notin \mathbb{N}\cup\{0\}$ we have*

$$\sum_{n=1}^{\infty} n^a = \sum_{n=1}^{\infty,\mathrm{Riesz}} n^a = \sum_{n=1}^{\infty,\mathrm{HK}} n^a + \delta_{a+1}\gamma = \sum_{n=1}^{\infty,\mathrm{HK}} n^a + \gamma\,\mathrm{Res}_0 \sum_{n=1}^{\infty} n^{a-z}.$$

Remark 2.19. For future use, it is useful to notice that all the regularised discrete sums, whether by cut-off, Riesz or heat-kernel procedures, coincide for homogeneous symbols with non-integer order.

3 Divergent integrals and sums of polyhomogeneous symbols

Polyhomogeneous symbols provide a natural extension of homogeneous functions insofar as they correspond to infinite sums of such functions with decreasing homogeneous degree. They encompass many examples of integrands in divergent integrals or sums that one encounters in mathematics and physics. Using the results of the previous section on regularised integrals and discrete sums of homogeneous functions, one defines regularised integrals and discrete sums of polyhomogeneous symbols.

3.1 Polyhomogeneous symbols. We only provide a few definitions and refer the reader to [36], [38], [39] for further details on classical pseudodifferential symbols. A further extension includes log-polyhomogeneous symbols, which allow for logarithmic as well as homogeneous components.

Let us denote by $\mathcal{S}(\mathbb{R}^d)$ the set of smooth complex valued functions on $\mathbb{R}^d$ called *symbols with constant coefficients* for which there exists a real number r such that for any multiindex $\beta \in \mathbb{N}^d$ there is a constant $C(\beta)$ satisfying the requirement

$$|\partial_\xi^\beta \sigma(\xi)| \le C(\beta)|\langle\xi\rangle^{r-|\beta|} \quad \text{for all } \xi \in \mathbb{R}^d, \tag{17}$$

where we have set $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ with $|\cdot|$ the Euclidean norm of ξ.

Let $\mathcal{S}^a(\mathbb{R}^d)$ be the subset of symbols for which one can choose r to be the real part $\mathrm{Re}(a)$ of a.

Remark 3.1. The presence of $\langle\xi\rangle$ takes care of infrared divergences, namely divergences for small values of $|\xi|$.

Note that $\mathcal{S}(\mathbb{R}^d) = \bigcup_{a\in\mathbb{C}} \mathcal{S}^a(\mathbb{R}^d)$. We call *smoothing* any symbol in the set

$$\mathcal{S}^{-\infty}(\mathbb{R}^d) = \bigcap_{a\in\mathbb{C}} \mathcal{S}^a(\mathbb{R}^d).$$

Clearly, $\mathcal{S}^{-\infty}(\mathbb{R}^d)$ is a subset of the algebra $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions on $\mathbb{R}^d$.

"Equality modulo smoothing symbols" defined by

$$\sigma \sim \sigma' \iff \sigma - \sigma' \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$$

yields an equivalence relation in $\mathcal{S}^a(\mathbb{R}^d)$ for any complex number a. For symbols $\sigma \in \mathcal{S}^a(\mathbb{R}^s)$, $\sigma_k \in \mathcal{S}^{a_k}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, where a_k has decreasing real part as k tends to infinity and $a_0 = a$, we further set

$$\Big(\sigma \sim \sum_{k\in\mathbb{N}_0} \sigma_k\Big) \iff \Big(\text{for all } \alpha \in \mathbb{R} \text{ there exists } K(\alpha) \in \mathbb{N} \text{ such that}$$
$$K \geq K(\alpha) \implies \sigma - \sum_{k\leq K} \sigma_k \in \mathcal{S}^\alpha(\mathbb{R}^d)\Big).$$

We now focus on classical (or polyhomogeneous) and log-polyhomogeneous symbols with constant coefficients.

Definition 3.2. Let a be a complex number.

(1) A symbol $\sigma \in \mathcal{S}^a(\mathbb{R}^d)$ is *classical* (or *polyhomogeneous*) of order a if

$$\sigma(\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\sigma_{a-j}(\xi)$$

where

- χ is any smooth function on $\mathbb{R}^d$ such that χ vanishes in a small neighborhood of 0 and is identically one outside the unit ball,
- $\sigma_{a-j} \in C^\infty(\mathbb{R}^d)$ is positively homogeneous of order $a - j$, i.e.,

$$\sigma_{a-j}(t\xi) = t^{a-j}\sigma_{a-j}(\xi)$$

for any $t > 0$ and any $\xi \in \mathbb{R}^d$.

(2) A symbol $\sigma \in \mathcal{S}^a(\mathbb{R}^d)$ is *log-polyhomogeneous of log-type* k and order a for some non-negative integer k if

$$\sigma(\xi) \sim \sum_{l=0}^{k} \sum_{j=0}^{\infty} \chi(\xi)\,\sigma_{a-j,l}(\xi) \log^l |\xi|$$

with $\sigma_{a-j,l}$, $l = 0, \ldots, k$, positively homogeneous functions of order $a - j$.

Remark 3.3. One easily checks that this definition is independent of the choice of cut-off function χ obeying the above conditions.

Example 3.4. (1) The map $\xi \mapsto \langle \xi \rangle^{-2} = \frac{1}{|\xi|^2+1}$ defines a classical symbol of order -2.

(3) The map $\xi \mapsto \log(|\xi|^2+1)$ is log-polyhomogeneous of log-type 1 and order zero.

(2) More generally, any smooth rational function is a classical symbol and any k-th logarithmic power thereof is a log-polyhomogeneous symbol of log-type k.

Let $\mathrm{CS}^{a,k}(\mathbb{R}^d) \subset S(\mathbb{R}^d)$ denote the class of log-polyhomogeneous symbols of order a and log-type k and let us set $\mathrm{CS}^{a,*}(\mathbb{R}^d) := \bigcup_{k\in\mathbb{N}_0} \mathrm{CS}^{a,k}(\mathbb{R}^d)$. Then $\mathrm{CS}^{a}(\mathbb{R}^d) := \mathrm{CS}^{a,0}(\mathbb{R}^d)$ corresponds to the set of classical symbols of order a. For any complex number a, the set $\mathrm{CS}^{a}(\mathbb{R}^d)$ is a subset of $\mathcal{S}^a(\mathbb{R}^d)$. Furthermore, the intersection $\bigcap_{a\in\mathbb{C}} \mathrm{CS}^{a}(\mathbb{R}^d)$ coincides with the algebra $\mathcal{S}^{-\infty}(\mathbb{R}^d)$ of smoothing symbols. It is also useful to observe that $\mathrm{CS}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) = \mathrm{CS}^{<-d}(\mathbb{R}^d)$, where $\mathrm{CS}^{<p}(\mathbb{R}^d) := \bigcup_{\mathrm{Re}(a)<p} \mathrm{CS}^{a}(\mathbb{R}^d)$ stands for the set of classical symbols whose order has real part $< p$.

Definition 3.5. The component $\sigma^L := \sigma_a \in C^\infty(\mathbb{R}^d)$ of highest homogeneity degree of a classical symbol σ in $\mathrm{CS}^{a}(\mathbb{R}^d)$ is called the *leading symbol* of the symbol.

Note that

$$\sigma_a(\xi) = \lim_{t\to\infty} (t^{-a}\sigma(t\,\xi)) \quad \text{for all } \xi \in \mathbb{R}^d.$$

The ordinary product of functions sends $\mathrm{CS}^{a}(\mathbb{R}^d) \times \mathrm{CS}^{b}(\mathbb{R}^d)$ to $\mathrm{CS}^{a+b}(\mathbb{R}^d)$ provided $b-a \in \mathbb{Z}$; let

$$\mathrm{CS}(\mathbb{R}^d) = \langle \textstyle\bigcup_{a\in\mathbb{C}} \mathrm{CS}^{a}(\mathbb{R}^d) \rangle$$

denote the algebra generated by all classical symbols with constant coefficients on $\mathbb{R}^d$. Let us denote by

$$\mathrm{CS}^{\notin\mathbb{Z}}(\mathbb{R}^d) := \bigcup_{a\in\notin\mathbb{Z}} \mathrm{CS}^{a}(\mathbb{R}^d) \quad \text{and} \quad \mathrm{CS}^{\notin\mathbb{Z},k}(\mathbb{R}^d) := \bigcup_{a\in\notin\mathbb{Z}} \mathrm{CS}^{a,k}(\mathbb{R}^d)$$

for any non-negative integer k.

Let us finally equip the set $\mathrm{CS}^{a}(\mathbb{R}^d)$ of classical symbols of order a with a Fréchet structure with the help of the following semi-norms labelled by multi-indices β and integers $j \geq 0$, N (see [21], see also [33]):

$$\sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^{-\mathrm{Re}(a)+|\beta|} \, \|\partial_\xi^\beta \sigma(\xi)\|,$$

$$\sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^{-\mathrm{Re}(a)+N+|\beta|} \|\partial_\xi^\beta \big(\sigma - \sum_{j=0}^{N-1} \chi(\xi)\sigma_{a-j}\big)(\xi)\|,$$

$$\sup_{|\xi|=1} \|\partial_\xi^\beta \sigma_{a-j}(\xi)\|,$$

where χ is any smooth function which vanishes in a neighborhood of zero and is identically one outside the unit ball. We shall refer to this topology as the topology on symbols of constant order.

In this topology, we have

$$\sigma = \lim_{N\to\infty} \sum_{j=0}^{N-1} \chi\sigma_{a-j}, \tag{18}$$

which justifies the notation $\sigma \sim \sum_{j=0}^{\infty} \chi\sigma_{a-j}$.

3.2 Cut-off regularised integrals and sums of symbols. Given a symbol $\sigma \in \mathrm{CS}^a(\mathbb{R}^d)$ for some complex number a, we want to investigate the asymptotic behaviour of $\int_{B(0,R)} \sigma$ as R tends to infinity, where $B(0, R)$ is the ball of radius R centered at zero. The following splitting is useful in understanding various asymptotic behaviours at infinity:

$$\sigma = \sum_{j=0}^{N-1} \sigma_{a-j}\chi + \sigma_{(N)}^{\chi}, \tag{19}$$

where σ_{a-j} is positively homogeneous of degree $a-j$ and $\sigma_{(N)}^{\chi}$ is a classical symbol of order $a-N$, with χ a smooth cut-off function which vanishes in a neighborhood of 0 and which is identically one outside the open unit ball.

Remark 3.6. Given two smooth cut-off functions χ and χ' with the above properties, the difference $\sigma_{(N)}^{\chi} - \sigma_{(N)}^{\chi'}$ has support in the open unit ball.

Choosing N large enough so that $\mathrm{Re}(a)-N<-d$ one splits the integral $\int_{B(0,R)} \sigma\, d\xi$ according to (19) into a convergent and divergent part as R goes to infinity:

$$\int_{B(0,R)} \sigma = \sum_{j=0}^{N-1} \int_{B(0,R)} \sigma_{a-j}\chi + \int_{B(0,R)} \sigma_{(N)}^{\chi}.$$

There is a finite number of terms $\int_{|\xi|\leq R} \sigma_{a-j}\,\chi$ that diverge as R tends to infinity, which we need to investigate in further detail. Using the homogeneity of σ_{a-j} we write

$$\begin{aligned}\int_{B(0,R)} \sigma_{a-j}\chi &= \int_{B(0,1)} \sigma_{a-j}\,\chi + \int_{B(0,R)-B(0,1)} \sigma_{a-j}\\ &= \int_{B(0,1)} \sigma_{a-j}\chi + \left(\int_1^R r^{a+d-1-j}\,dr\right)\int_{S^{d-1}} \sigma_{a-j}(\omega)\,d_S\omega,\end{aligned}$$

where $d_S\omega$ is the volume measure on the unit sphere S^{d-1} induced by the canonical volume measure on $\mathbb{R}^d$. Thus, divergences of $\int_{B(0,R)} \sigma$ as R tends to infinity arise from those of $\int_1^R r^{a+d-1-j}\,dr$ studied previously. We therefore define the regularised cut-off integral of σ in terms of the regularised cut-off integral $⨍_1^{\infty} r^{a+d-1-j}\,dr$.

Definition 3.7. Let a be a complex number. The cut-off integral of a symbol σ in $\mathrm{CS}^a(\mathbb{R}^d)$ is defined by

$$\begin{aligned} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma := & \sum_{j=0}^{N-1} \left(-\!\!\!\!\!\!\int_1^\infty r^{a+d-1-j}\, dr \right) \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega \\ & + \sum_{j=0}^{N-1} \int_{B(0,1)} \sigma_{a-j}\,\chi + \int_{R^d} \sigma_{(N)}^\chi . \end{aligned} \tag{20}$$

A similar definition holds for log-polyhomogeneous symbols, which we do not give here, referring the reader to e.g. [24], [31].

Remark 3.8. The expression $\sum_{j=0}^{N-1} \int_{B(0,1)} \sigma_{a-j}\, \chi + \int_{R^d} \sigma_{(N)}^\chi$ is independent of χ due to the fact that given another smooth cut-off function χ' with the same properties, the difference $\chi - \chi'$ has support in the unit ball $B(0,1)$.

Cut-off regularised sums of classical symbols on $\mathbb{R}$ are defined in a similar manner. Given a symbol σ in $\mathrm{CS}(\mathbb{R})$ using the fact that the cut-off function is equal to one outside $]-1,1[$, using (19) we split the sum $\sum_{n=1}^N \sigma(n)$ into the following convergent and divergent parts as N goes to infinity:

$$\sum_{n=1}^N \sigma(n) = \sum_{j=0}^{N-1} \sum_{n=1}^N \sigma_{a-j}(n) + \sum_{n=1}^N \sigma_{(N)}^\chi(n).$$

Remark 3.9. Given another smooth cut-off function χ' with the same properties as the function χ, the difference $\sigma_{(N)}^\chi - \sigma_{(N)}^{\chi'}$ has support in $]-1,1[$, so that $\sum_{n=1}^N \sigma_{(N)}^\chi = \sum_{n=1}^N \sigma_{(N)}^{\chi'}$. Thus, the sum $\sum_{n=1}^N \sigma_{(N)}^\chi$ is independent of χ and we henceforth denote it by $\sum_{n=1}^N \sigma_{(N)}$.

There is a finite number of terms $\sum_{n=1}^N \sigma_{a-j}(n)$ that diverge as N tends to infinity, which we need to investigate in further detail. Using the homogeneity of σ_{a-j} as before we write

$$\sum_{n=1}^N \sigma_{a-j}(n) = \Big(\sum_{n=1}^N n^{a+d-1-j} \Big) \sigma_{a-j}(1).$$

We therefore define the regularised cut-off sum of σ over positive integers in terms of the regularised cut-off sum $-\!\!\!\!\!\!\sum_{n=1}^\infty n^{a+d-1-j}$.

Definition 3.10. The cut-off sum over positive integers of a symbol σ in $\mathrm{CS}(\mathbb{R})$ of complex order a is defined by

$$-\!\!\!\!\!\!\sum_{n=1}^\infty \sigma(n) := \sum_{j=0}^{N-1} \Big(-\!\!\!\!\!\!\sum_{n=1}^\infty n^{a+d-1-j} \Big) \sigma_{a-j}(1) + \sum_{n=1}^\infty \sigma_{(N)}^\chi(n). \tag{21}$$

3.3 Holomorphic regularisation. A family $\{f(z)\}_{z\in\Omega}$ in a topological vector space E parametrised by a complex domain Ω is *holomorphic* at $z_0 \in \Omega$ if the corresponding function $f:\Omega \to E$ is uniformly complex-differentiable on compact subsets in a neighborhood of z_0. When f takes its values in a Banach space E the existence of a complex derivative implies that a holomorphic function is actually infinitely differentiable and admits a Taylor expansion in a neighborhood N_{z_0} of z_0 (see e.g. [20], Theorem 8.1.7, or [12], Section XV.5.1). We shall take for granted that this analyticity property of holomorphic functions actually extends to any inductive limit of Fréchet spaces and can therefore be implemented in our context (see [29] for a more detailed discussion on this point).

Definition 3.11. Let Ω be a domain of $\mathbb{C}$. A family $(\sigma(z))_{z\in\Omega}$ of order $(\alpha(z))_{z\in\Omega}$ is a holomorphic family of classical symbols in $\mathrm{CS}(\mathbb{R}^d)$ parametrised by Ω when

(1) the map $z \mapsto \alpha(z)$ with $\alpha(z)$ the order of $\sigma(z)$, is holomorphic in z,

(2) $z \mapsto \sigma(z)$ is holomorphic as element of $C^\infty(\mathbb{R}^d)$ and for each $z \in \Omega$, $\sigma(z) \sim \sum_{j=0}^{\infty} \chi\sigma(z)_{\alpha(z)-j}$ (for some smooth function χ which is identically one outside the open unit ball and vanishes in a neighborhood of 0) lies in $\mathrm{CS}^{\alpha(z)}(\mathbb{R}^d)$,

(3) for any positive integer N, the remainder term

$$\sigma_{(N)}(z) = \sigma(z) - \sum_{j=0}^{N-1} \sigma(z)_{\alpha(z)-j}$$

is holomorphic in $z \in \Omega$ as an element of $C^\infty(\mathbb{R}^d)$ and its k-th derivative $\partial_z^k \sigma_{(N)}(z)$ lies in $S^{\alpha(z)-N+\varepsilon}(U)$ for all $\varepsilon > 0$ locally uniformly in z, i.e., the k-th derivative $\partial_z^k \sigma_{(N)}(z)$ satisfies a uniform estimate (17) in z on compact subsets in Ω.

In particular, for any integer $j \geq 0$, the (positively) homogeneous component $\sigma_{\alpha(z)-j}(z)$ of degree $\alpha(z) - j$ of the symbol is holomorphic on Ω as an element of $C^\infty(\mathbb{R}^n)$.

Remark 3.12. Differentiation in the parameter z does not modify the order (see Lemma 1.16 in [32]).

Definition 3.13. We call *holomorphic regularisation* a linear map

$$\mathcal{R}: \sigma \mapsto \sigma(z)$$

which sends a symbol σ in $\mathrm{CS}(\mathbb{R}^d)$ to a holomorphic family $\sigma(z)$ of non-constant affine order $\alpha(z) = a - qz$ in $\mathrm{CS}(\mathbb{R}^d)$, where a is the order of $\sigma = \sigma(0)$ and q a non-zero real constant.

Example 3.14. *Riesz regularisations*[11] are holomorphic regularisations of the type

$$\mathcal{R}(\sigma)(z)(\xi) = \sigma(\xi)|\xi|^{-z} \quad \text{for all } |\xi| \geq 1$$

[11]Also called modified dimensional regularisation in the physics literature; see e.g. [35], Example 3, chapter II, for a mathematical presentation.

and we shall also consider the slightly more general holomorphic regularisations of the type

$$\mathcal{R}(\sigma)(z)(\xi) = H(z)\sigma(\xi)|\xi|^{-z} \quad \text{for all } |\xi| \geq 1,$$

with H holomorphic such that $H(0) = 1$. The case $H(z) := \frac{\mathrm{Vol}(S^{d-z-1})}{\mathrm{Vol}(S^{d-1})}$ where we have set $\mathrm{Vol}(S^{d-z-1}) := \frac{2\pi^{\frac{d-z}{2}}}{\Gamma\left(\frac{d-z}{2}\right)}$ and which, at $z = 0$, coincides with the volume of the unit sphere in d dimensions, corresponds to *dimensional regularisation* commonly used in perturbative quantum field theory.

Substituting $\sigma(z)$ for σ and $a - qz$ for a in (20) yields

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z) := \sum_{j=0}^{N-1} \left(-\!\!\!\!\!\!\int_1^\infty r^{a-qz+d-1-j}\, dr\right) \int_{S^{d-1}} \sigma_{a-j}(z)(\omega)\, d_S\omega \\
+ \sum_{j=0}^{N-1} \int_{B(0,1)} \sigma_{a-j}(z)\, \chi + \int_{R^d} \sigma_{(N)}^{\chi}(z).
\end{aligned}$$

Provided the integer N is chosen large enough, this defines a meromorphic function on the whole complex plane, with a simple pole at zero expressed in terms of that of $\sum_{j=0}^{N-1} -\!\!\!\!\!\!\int_1^\infty r^{a+d-q\,z-1-j}\, dr$ for the other terms all involve holomorphic functions at zero. It actually defines a meromorphic extension $-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)$ to the whole complex plane, with simple poles in $\{\frac{a+d-k}{q},\ k \in \mathbb{Z}_+\}$, of the holomorphic map $\int_{\mathbb{R}^d} \sigma(z)$ defined on $\mathrm{Re}(z) > \frac{\mathrm{Re}(a)+d}{q}$. The residue at $z = 0$ reads

$$\begin{aligned}
\mathrm{Res}_0\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)\right) &= \sum_{j=0}^{N-1} \mathrm{Res}_0\left(-\!\!\!\!\!\!\int_1^\infty r^{a+d-qz-1-j}\, dr\right) \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega \\
&= \sum_{j=0}^{N-1} \frac{\delta_{a+d-j}}{q} \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega.
\end{aligned}$$

Picking the finite part at $z = 0$ leads to the following definition.

Definition 3.15. Given a complex number a and a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto$

$\sigma(z)$ on $\mathrm{CS}(\mathbb{R}^d)$, the $\mathcal{R}$-*regularised integral* of a symbol σ in $\mathrm{CS}^a(\mathbb{R}^d)$ is defined by

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma &:= \sum_{j=0}^{N-1} \left(-\!\!\!\!\!\!\int_1^\infty r^{a+d-1-j}\, dr \right) \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega \\
&\quad + \sum_{j=0}^{N-1} \frac{\delta_{a+d-j}}{q} \int_{S^{d-1}} \sigma'_{a-j}(0)(\omega)\, d_S\omega \\
&\quad + \sum_{j=0}^{N-1} \int_{B(0,1)} \sigma_{a-j}\,\chi + \int_{R^d} \sigma^{\chi}_{(N)} \\
&= -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma + \sum_{j=0}^{N-1} \frac{\delta_{a+d-j}}{q} \int_{S^{d-1}} \sigma'_{a-j}(0)(\omega)\, d_S\omega.
\end{aligned} \tag{22}$$

Remark 3.16. The dependence on the regularisation $\mathcal{R}$ only arises in the term

$$\sum_{j=0}^{N-1} \frac{\delta_{a+d-j}}{q} \int_{S^{d-1}} \sigma'_{a-j}(0)(\omega)\, d_S\omega.$$

The occurrence of a derivative $\sigma'(0)$ is a special instance of more general results of [32]. When the order a of σ (which is also the order of $\sigma'(0)$) is non-integer, then the additional term vanishes and we have

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma$$

independently of the chosen regularisation $\mathcal{R}$.

$\mathcal{R}$-regularised discrete sums are defined similarly. Substituting $\sigma(z)$ to σ in (21) yields

$$\sum_{n=1}^{\infty} \sigma(z)(n) := \sum_{j=0}^{N-1} \Big(\sum_{n=1}^{\infty} n^{a-q\,z-j} \Big) \sigma_{a-j}(z)(1) + \sum_{n=1}^{\infty} \sigma^{\chi}_{(N)}(z)(n).$$

This defines a meromorphic extension $\sum_{n=1}^{\infty} \sigma(z)(n)$ to the whole complex plane with simple poles in $\left\{\frac{a+1-k}{q},\, k \in \mathbb{Z}_+\right\}$ of the holomorphic map $\sum_{n=1}^{\infty} \sigma(z)(n)$ defined on the half plane $\mathrm{Re}(z) > \frac{\mathrm{Re}(a)+1}{q}$. The pole at zero reads

$$\mathrm{Res}_0 \Big(\sum_{n=1}^{\infty} \sigma(z)(n) \Big) = \sum_{j=0}^{N-1} \mathrm{Res}_0 \left(-\!\!\!\!\!\!\int_1^\infty n^{a-q\,z-j} \right) \sigma_{a-j}(1) = \sum_{j=0}^{N-1} \frac{\delta_{a+1-j}}{q} \sigma_{a-j}(1).$$

This leads to the following definition.

Definition 3.17. Let a be a complex number. Given a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R})$, the *$\mathcal{R}$-regularised discrete sum* on positive integers of a symbol σ in $\mathrm{CS}^a(\mathbb{R})$ is defined by

$$\begin{aligned}\sum_{n=1}^{\infty,\mathcal{R}} \sigma(n) &:= \sum_{j=0}^{N-1} \mathrm{ev}_0^{\mathrm{reg}}\Big(\sum_{n=1}^{\infty} n^{a-q\,z-j}\sigma_{a-j}(z)(1)\Big) + \sum_{n=1}^{\infty} \sigma_{(N)}^{\chi}(n) \\ &= \sum_{j=0}^{N-1} \sigma_{a-j}(1) \sum_{n=1}^{\infty,\mathrm{Riesz}} n^{a-j} + \sum_{j=0}^{N-1} \frac{\delta_{a+1-j}}{q}\sigma'_{a-j}(0)(1) + \sum_{n=1}^{\infty} \sigma_{(N)}^{\chi}(n).\end{aligned} \tag{23}$$

Remark 3.18. The dependence on the regularisation $\mathcal{R}$ only arises in the term $\sum_{j=0}^{N-1} \frac{\delta_{a+1-j}}{q}(\sigma'(0))_{a-j}(1)$. When the order a of σ (which is also the order of $\sigma'(0)$) is non-integer, then $\sum_{n=1}^{\infty,\mathrm{Riesz}} n^{a-j} = \sum_{n=1}^{\infty} n^{a-j}$ so that

$$\sum_{n=1}^{\infty,\mathcal{R}} \sigma(n) = \sum_{j=0}^{N-1} \sum_{n=1}^{\infty,\mathrm{Riesz}} n^{a-j}\sigma_{a-j}(1) + \sum_{n=1}^{\infty} \sigma_{(N)}^{\chi}(n)$$

is independent of the chosen regularisation $\mathcal{R}$.

3.4 Heat-kernel regularised integrals and sums of symbols. Let $\sigma \in \mathrm{CS}(\mathbb{R}^d)$ be a symbol of complex order a. For any $\varepsilon > 0$ the integral $\int_{\mathbb{R}^d} \sigma(\xi)\, e^{-\varepsilon|\xi|^2} d\xi$ is well defined. Using the splitting (19) we rewrite it as a sum of convergent terms as $\varepsilon \to 0$ and a possibly divergent term involving $\int_1^{\infty} r^{a+d-j-1} e^{-\varepsilon r^2} dr$:

$$\begin{aligned}\int_{\mathbb{R}^d} \sigma(\xi) e^{-\varepsilon|\xi|^2} d\xi &= \sum_{j=0}^{N-1} \int_{\mathbb{R}^d} \sigma_{a-j}(\xi)\chi(\xi) e^{-\varepsilon|\xi|^2} d\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(\xi) e^{-\varepsilon|\xi|^2} d\xi \\ &= \sum_{j=0}^{N-1} \int_{B(0,1)} \sigma_{a-j}(\xi)\chi(\xi) e^{-\varepsilon|\xi|^2} d\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(\xi) e^{-\varepsilon|\xi|^2} d\xi \\ &\quad + \sum_{j=0}^{N-1} \left(\int_1^{\infty} r^{a+d-j-1} e^{-\varepsilon r^2} dr\right) \int_{S^{d-1}} \sigma_{a-j}(\omega) e^{-\varepsilon} d_S\omega.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0} \int_{B(0,1)} \sigma_{a-j}(\xi)\chi(\xi) e^{-\varepsilon|\xi|^2} d\xi = \int_{B(0,1)} \sigma_{a-j}(\xi)\chi(\xi)\, d\xi$$

and

$$\lim_{\varepsilon\to 0} \int_{S^{d-1}} \sigma_{a-j}(\omega) e^{-\varepsilon} d_S\omega = \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega,$$

taking finite part as ε tends to zero in the expression $\int_{\mathbb{R}^d} \sigma(\xi) e^{-\varepsilon|\xi|^2} d\xi$ leads to the following definition.

Definition 3.19. The *heat-kernel regularised integral* of a symbol σ in $\mathrm{CS}^a(\mathbb{R}^d)$ is defined by

$$\begin{aligned}\int_{\mathbb{R}^d}^{\mathrm{HK}} \sigma(\xi)\, d\xi := \int_{B(0,1)} \sigma_{a-j}(\xi)\chi(\xi)\, d\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(\xi)\, d\xi \\ + \sum_{j=0}^{N-1} \left(\int_1^{\infty,\mathrm{HK}} r^{a+d-j-1} dr \right) \int_{S^{d-1}} \sigma_{a-j}(\omega)\, d_S\omega,\end{aligned}$$

where $\int_1^{\infty,\mathrm{HK}} r^{a+d-j-1}\, dr$ is the heat-kernel regularised integral of a homogeneous function previously defined.

Remark 3.20. Cut-off, $\mathcal{R}$-holomorphic and heat-kernel regularised integrals coincide on non-integer order symbols.

A similar procedure leads to heat-kernel regularised sums. Let $\sigma \in \mathrm{CS}(\mathbb{R})$ be a symbol of complex order a. For any $\varepsilon > 0$ the discrete sum $\sum_{n=1}^{\infty} \sigma(n)e^{-\varepsilon n}$ is well defined. Using the splitting (19) we rewrite it as a sum of convergent terms as ε tends to 0 and a possibly divergent term involving $\sum_{n=1}^{\infty} n^{a-j} e^{-\varepsilon n}$:

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma(n)e^{-\varepsilon n} &= \sum_{j=0}^{N-1} \sum_{n=1}^{\infty} \sigma_{a-j}(n)e^{-\varepsilon n} + \sum_{n=1}^{\infty} \sigma_{(N)}(n)e^{-\varepsilon n} \\ &= \sum_{n=1}^{\infty} \sigma_{(N)}(n)e^{-\varepsilon n} + \sum_{j=0}^{N-1} \Big(\sum_{n=1}^{\infty} r^{a-j} e^{-\varepsilon n}\Big)\sigma_{a-j}(1)e^{-\varepsilon}.\end{aligned}$$

For a large enough integer N,

$$\lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} \sigma_{(N)}(n)e^{-\varepsilon n} = \sum_{n=1}^{\infty} \sigma_{(N)}(n)$$

so that taking finite part as ε tends to zero in the expression $\sum_{n=1}^{\infty} \sigma(n)e^{-\varepsilon n}$ leads to the following definition.

Definition 3.21. The *heat-kernel regularised discrete sum* of a symbol σ in $\mathrm{CS}^a(\mathbb{R})$ is defined by

$$\sum_{n=1}^{\infty,\mathrm{HK}} \sigma(n) := \sum_{n=1}^{\infty} \sigma_{(N)}(n) + \sum_{j=0}^{N-1} \Big(\sum_{n=1}^{\infty,\mathrm{HK}} n^{a-j}\Big)\sigma_{a-j}(1),$$

where $\sum_{n=1}^{\infty,\mathrm{HK}} n^{a-j}$ is the heat-kernel regularised sum of a homogeneous function previously defined.

Remark 3.22. Cut-off, $\mathcal{R}$-holomorphic and heat-kernel regularised discrete sum all coincide on non-integer order symbols.

4 Regularised integrals and discrete sums of tensor products of symbols

Regularised integrals and discrete sums naturally extend in a multiplicative way (with respect to the tensor product) to tensor algebras of symbols; in the algebraic setup described in the Incipit, this corresponds to building a character $\mathrm{T}(\lambda)$ from the linear form λ given by either a regularised integral or a regularised discrete sum. There are nevertheless various ways of doing so, which, even if artificial in this context, are interesting to work out in view of the more ambitious extensions involving constraints that we have in mind. Tensor products therefore offer a useful toy model to compare the evaluator and Birkhoff–Hopf factorisation methods when evaluating multiple integrals and multiple sums.

4.1 A canonical extension of regularised integrals and sums. The following elementary result, already mentioned in the Incipit in a purely algebraic setup, is useful to show that regularised integrals on $\mathrm{CS}(\mathbb{R}^d)$, resp. regularised sums on $\mathrm{CS}(\mathbb{R})$, canonically extend to the tensor algebra $\mathcal{T}(\mathrm{CS}(\mathbb{R}^d))$ generated by $\mathrm{CS}(\mathbb{R}^d)$ resp. $\mathcal{T}(\mathrm{CS}(\mathbb{R}))$ generated by $\mathrm{CS}(\mathbb{R})$.

By the universal property of tensor algebras, any (continuous) linear form $L\colon V \to \mathbb{C}$ on a (topological) vector space V extends in a unique way to a (continuous) character $\widetilde{L}\colon \mathcal{T}(V) \to \mathbb{C}$, i.e., such that for all $v_1, \dots, v_{k+l}$ in V

$$\widetilde{L}((v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_{k+l})) = \widetilde{L}(v_1 \otimes \cdots \otimes v_k) \cdot \widetilde{L}(v_{k+1} \otimes \cdots \otimes v_{k+l}).$$

Here $\mathcal{T}(V)$ is the (Grothendieck closure $\bigoplus_{k=0}^{\infty} \hat{\otimes}^k V$ of the) tensor algebra $\bigoplus_{k=0}^{\infty} \otimes^k V$.

In this way, one canonically extends regularised sums and integrals of symbols to regularised multiple integrals and sums of tensor products of symbols. Let us first check their continuity properties. The continuity of $⨍$, resp. $\sum$, can be seen on direct inspection of formulae (20) resp. (21). Similarly, given a holomorphic regularisation $\mathcal{R}$ on $\mathrm{CS}(\mathbb{R}^d)$, resp. on $\mathrm{CS}(\mathbb{R})$, the continuity of $⨍^{\mathcal{R}}$ resp. $\sum^{\mathcal{R}}$ can be seen on direct inspection of formulae (22) resp. (23). This leads to the following statement.

Proposition 4.1. *Given a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R}^n)$ (resp. on $\mathrm{CS}(\mathbb{R})$), the regularised integrals $⨍$ and $⨍^{\mathcal{R}}$ (resp. the regularised discrete sums $\sum$ and $\sum^{\mathcal{R}}$) extend in a unique way to a **character** on the tensor algebras $\mathcal{T}(\mathrm{CS}(\mathbb{R}^d))$ (resp. $\mathcal{T}(\mathrm{CS}(\mathbb{R}))$) by*

$$⨍_{\mathbb{R}^{kd}}^{\mathrm{ren}} \bigotimes_{i=1}^k \sigma_i = \prod_{i=1}^k ⨍_{\mathbb{R}^d} \sigma_i \quad (\textit{resp.}\ \sum_{\mathbb{N}^k}^{\mathrm{ren}} \bigotimes_{i=1}^k \sigma_i = \prod_{i=1}^k \sum_N \sigma_i)$$

and similarly

$$⨍_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{ren}} \bigotimes_{i=1}^k \sigma_i = \prod_{i=1}^k ⨍_{\mathbb{R}^d}^{\mathcal{R}} \sigma_i \quad (\textit{resp.}\ \sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{ren}} \bigotimes_{i=1}^k \sigma_i = \prod_{i=1}^k \sum_N^{\mathcal{R}} \sigma_i).$$

Alternatively, given a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R}^d)$ (resp. $\mathrm{CS}(\mathbb{R})$) and symbols $\sigma_1, \dots, \sigma_k$ in $\mathrm{CS}(\mathbb{R}^d)$ (resp. in $\mathrm{CS}(\mathbb{R})$), one can replace σ_i in the above construction by $\sigma_i(z_i)$. This gives rise to a meromorphic map in several variables

$$(z_1, \dots, z_k) \mapsto \prod_{i=1}^{k} \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma_i(z_i) \quad \text{(resp. } (z_1, \dots, z_k) \mapsto \prod_{i=1}^{k} \sum_{\mathbb{N}} \sigma_i(z_i)),$$

i.e., to maps in $\mathcal{T}\left(\mathrm{Mer}_0^1(\mathbb{C})\right)$ with the notation of the first section. We can implement any generalised evaluator Λ. As we saw they all give rise to

$$\Lambda\Big(\prod_{i=1}^{k} \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma_i(z_i)\Big) = \prod_{i=1}^{k} \lambda\Big(\int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma_i(z_i)\Big)$$

$$\text{(resp. } \Lambda\Big(\prod_{i=1}^{k} \sum_{n=1}^{\infty} \sigma_i(z_i)(n)\Big) = \prod_{i=1}^{k} \lambda\Big(\sum_{n=1}^{\infty} \sigma_i(z_i)(n)\Big)),$$

where $\lambda := \Lambda_1$ is a regularised evaluator at zero. Choosing $\lambda := \mathrm{ev}_0^{\mathrm{reg}}$ yields back the canonical extension of the $\mathcal{R}$-integral (resp. $\mathcal{R}$-sum) introduced above:

$$\Lambda\Big(\prod_{i=1}^{k} \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma_i(z_i)\Big) = \prod_{i=1}^{k} \int\!\!\!\!\!\!-_{\mathbb{R}^d}^{\mathcal{R}} \sigma_i \quad \text{(resp. } \Lambda\Big(\prod_{i=1}^{k} \sum_{n=1} \sigma_i(z_i)(n)\Big) = \prod_{i=1}^{k} \sum_{n=1}^{\infty, \mathcal{R}} \sigma_i(n)).$$

4.2 An extension by Birkhoff–Hopf factorisation. Yet another method to extend the $\mathcal{R}$-integral (resp. $\mathcal{R}$-sum) to tensor products is to set $z_i = z$ for $i \in \{1, \dots, I\}$ and to implement a renormalisation procedure using Birkhoff–Hopf factorisation which was initiated by Connes and Kreimer [8], [23] (see [25] for a mathematical presentation) and inspired by the renormalisation techniques used in perturbative renormalisation. This approach, which is rather artificial for the toy model under consideration, is nevertheless instructive in view of later generalisations; we shall see that it yields back the previous renormalised multiple integrals and sums, which is not the case in the presence of linear or conical constraints which we investigate later in these notes. This presentation is based on joint work with Dominique Manchon [26], [27].

Remark 4.2. An identification of the parameters $z_i = z$ occurs in physics when using dimensional regularisation, for the parameter z used to complexify the dimension thereby modifies the integrands via a common complex parameter z.

For this purpose, one equips the tensor algebra $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V)$ of a linear space V, where we have set $\mathcal{T}^k(V) := \hat{\otimes}^k V$, with the ordinary tensor product $\otimes$ and the deconcatenation coproduct:

$$\Delta\colon \mathcal{T}(V) \to \bigoplus_{p+q=L} (\mathcal{T}^p(V) \otimes \mathcal{T}^q(V)),$$

$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \sum_{\{i_1,\dots,i_{l'}\} \subset \{1,\dots,k\}} (\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_{k'}}) \otimes (\sigma_{i_{l+1}} \otimes \cdots \otimes \sigma_{i_k}),$$

where $\{i_{k'+1},\dots,i_k\}$ is the complement in $\{1,\dots,k\}$ of the set $\{i_1,\dots,i_{k'}\}$. It is the unique algebra morphism from $\mathcal{T}(V)\to\mathcal{T}(V)\otimes\mathcal{T}(V)$ such that $\Delta(1)=1\otimes 1$ and $\Delta(x)=x\otimes 1+1\otimes x$ (see e.g. [25]).

Lemma 4.3. *The tensor algebra* $(\mathcal{T}(V),\otimes,\Delta)$ *of a linear space* V *is a graded* (*by the natural grading on tensor products*) *cocommutative connected Hopf algebra.*

Proof. We use Sweedler's notations and write in a compact form

$$\Delta\sigma=\sum_{(\sigma)}\sigma_{(1)}\otimes\sigma_{(2)}.$$

- The coproduct Δ is clearly compatible with the filtration.
- The coproduct Δ is cocommutative, for we have $\tau_{12}\circ\Delta=\Delta$, where τ_{ij} is the flip on the i-th and j-th entries:

$$\tau_{12}\circ\Delta(\sigma)=\tau_{12}\Big(\sum_{(\sigma)}\sigma_{(1)}\otimes\sigma_{(2)}\Big)=\sum_{(\sigma)}\sigma_{(2)}\otimes\sigma_{(1)}=\Delta(\sigma).$$

- The coproduct Δ is coassociative since

$$(\Delta\otimes 1)\circ\Delta(\sigma)=\sum_{(\sigma)}(\sigma_{(1:1)}\otimes\sigma_{(1:2)})\otimes\sigma_{(2)}=\sum_{(\sigma)}\sigma_{(1)}\otimes(\sigma_{(2:1)}\otimes\sigma_{(2:2)})=(1\otimes\Delta)\circ\Delta(\sigma).$$

- The co-unit ε defined by $\varepsilon(1)=1$ is an algebra morphism.
- The coproduct Δ is compatible with the product $\otimes$:

$$\begin{aligned}\Delta\circ m(\sigma\otimes\sigma')&=\sum_{(\sigma\otimes\sigma')}(\sigma\otimes\sigma')_{(1)}\otimes(\sigma\otimes\sigma')_{(2)}\\&=(m\otimes m)\circ\tau_{23}\circ[(\sigma_{(1)}\otimes\sigma_{(2)})\otimes(\sigma'_{(1)}\otimes\sigma'_{(2)})]\\&=(m\otimes m)\circ\tau_{23}\circ(\Delta\otimes\Delta)(\sigma\otimes\sigma').\end{aligned}$$ □

Applying these results to $V=\mathrm{CS}(\mathbb{R}^d)$ yields a cocommutative and connected Hopf algebra $\mathcal{H}_d:=\mathcal{T}(\mathrm{CS}(\mathbb{R}^d))$.

A continuous holomorphic regularisation procedure $\mathcal{R}\colon\sigma\mapsto(z\mapsto\sigma(z))$ on $\mathrm{CS}(\mathbb{R}^d)$ induces one on $\mathcal{T}(\mathrm{CS}(\mathbb{R}^d))$ defined by

$$\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)(z)=\sigma_1(z)\otimes\cdots\otimes\sigma_k(z),$$

which is compatible with the tensor product. For any σ_i, $i=1,\dots,k+k'$, in $\mathrm{CS}(\mathbb{R}^d)$ we have

$$\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k\otimes\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+k'})(z)=\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)\widetilde{\mathcal{R}}(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+k'}).$$

Let a_i be the order of $\sigma_i=\sigma_i(0)$. For $\mathrm{Re}(z)$ sufficiently large the real part of the order a_i-qz of each $\sigma_i(z)$ is sufficiently small for the following integrals and sums to converge and the following identities to hold:

$$\int_{(\mathbb{R}^d)^k}\sigma_1(z)\otimes\cdots\otimes\sigma_k(z)=\prod_{i=1}^k\int_{\mathbb{R}^d}\sigma_i(z),\quad\sum_{\mathbb{N}^k}\sigma_1(z)\otimes\cdots\otimes\sigma_k(z)=\prod_{i=1}^k\sum_{\mathbb{N}}\sigma_i(z).$$

By an analytic continuation argument, these multiplicativity properties extend beyond the domain of convergence, which leads to the following algebra morphisms:

$$\Phi\colon \mathcal{T}(\mathrm{CS}(\mathbb{R}^d)) \to \mathrm{Mer}_0(\mathbb{C}), \quad \sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \left(z \mapsto {\textstyle -\!\!\!\!\int_{(\mathbb{R}^d)^k}} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z)\right),$$

and

$$\Psi\colon \mathcal{T}(\mathrm{CS}(\mathbb{R})) \to \mathrm{Mer}_0(\mathbb{C}), \quad \sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \left(z \mapsto {\textstyle\sum_{\mathbb{N}^k}} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z)\right).$$

A Birkhoff–Hopf factorisation procedure then yields a complex valued character.

Theorem 4.4. *A continuous holomorphic regularisation procedure $\mathcal{R}$ on $\mathrm{CS}(\mathbb{R}^n)$ gives rise to a character on the Hopf algebra $(\mathcal{H}_d, \otimes)$*

$$\phi^{\mathcal{R}}\colon \mathcal{T}(\mathrm{CS}(\mathbb{R}^d)) \to \mathbb{C}, \quad \sigma_1 \otimes \cdots \otimes \sigma_k \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}} \sigma_1 \otimes \cdots \otimes \sigma_k,$$

and on the Hopf algebra $(\mathcal{H}_1, \otimes)$

$$\psi^{\mathcal{R}}\colon \mathcal{T}(\mathrm{CS}(\mathbb{R})) \to \mathbb{C}, \quad \sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{BF}} \sigma_1 \otimes \cdots \otimes \sigma_k.$$

In particular the following multiplicative properties hold. For any $\sigma_1, \ldots, \sigma_{k+k'}$ in $\mathrm{CS}(\mathbb{R}^d)$,

$$\begin{aligned} &-\!\!\!\!\!\!\int_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \otimes (\sigma'_{k+1} \otimes \cdots \otimes \sigma'_{k'}) \\ &\qquad = \left(-\!\!\!\!\!\!\int_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}} \sigma_1 \otimes \cdots \otimes \sigma_k\right) \cdot \left(-\!\!\!\!\!\!\int_{\mathbb{R}^{dk'}}^{\mathcal{R},\mathrm{BF}} \sigma'_{k+1} \otimes \cdots \otimes \sigma'_{k'}\right) \end{aligned}$$

and for any $\sigma_1, \ldots, \sigma_{k+k'}$ in $\mathrm{CS}(\mathbb{R})$

$$\begin{aligned} &\sum_{\mathbb{N}^{k+k'}}^{\mathcal{R},\mathrm{BF}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \otimes (\sigma'_{k+1} \otimes \cdots \otimes \sigma'_{k'}) \\ &\qquad = \left(\sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{BF}} \sigma_1 \otimes \cdots \otimes \sigma_k\right) \cdot \left(\sum_{\mathbb{N}^{k'}}^{\mathcal{R},\mathrm{BF}} \sigma'_{k+1} \otimes \cdots \otimes \sigma'_{k'}\right). \end{aligned}$$

Consequently,

$$-\!\!\!\!\!\!\int_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}} = -\!\!\!\!\!\!\int_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{ren}} \quad \textit{and} \quad \sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{BF}} = \sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{ren}}$$

so that the Birkhoff–Hopf factorisation procedure yields back the canonical multiplicative extension to tensor products described above.

Proof. Birkhoff–Hopf factorisation combined with a minimal subtraction scheme yields the existence of morphisms

$$\Phi_+^{\mathcal{R}}\colon (\mathcal{H}_d, \otimes) \to \mathrm{Hol}_0(\mathbb{C}) \quad \text{and} \quad \Psi_+^{\mathcal{R}}\colon (\mathcal{H}_1, \otimes) \to \mathrm{Hol}_0(\mathbb{C})$$

on the connected filtered commutative Hopf algebra $\mathcal{H}_d$ resp. $\mathcal{H}_1$ (see [25], Theorem II.5.1), which corresponds to the holomorphic part in the unique Birkhoff–Hopf decomposition

$$\Phi^{\mathcal{R}} = (\Phi_-^{\mathcal{R}})^{*-1} \star \Phi_+^{\mathcal{R}} \quad \text{and} \quad \Psi^{\mathcal{R}} = (\Psi_-^{\mathcal{R}})^{*-1} \star \Psi_+^{\mathcal{R}}$$

of $\Phi^{\mathcal{R}}$ and $\Psi^{\mathcal{R}}$. Here $\star$ stands for the convolution product on the Hopf algebra $\mathcal{H}_d$ resp. $\mathcal{H}_1$. The values $\phi^{\mathcal{R}} := \Phi_+^{\mathcal{R}}(0)$ and $\psi^{\mathcal{R}} := \Psi_+^{\mathcal{R}}(0)$ at $z = 0$ in turn yield characters

$$\phi^{\mathcal{R}}\colon (\mathcal{H}^0, \otimes) \to \mathbb{C}, \quad \psi^{\mathcal{R}}\colon (\mathcal{H}^0, \otimes) \to \mathbb{C},$$

which extend the maps given by the ordinary iterated integration and ordinary iterated summation. The multiplicativity of these renormalised integrals $⨍_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}}$ and $\sum_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}}$ with respect to tensor products follows from the character property of $\phi^{\mathcal{R}}$ and $\psi^{\mathcal{R}}$. By the universal property of the tensor algebra which leads to the uniqueness of the extension of a character to the tensor algebra, the character $⨍_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{BF}}$ (resp. $\sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{BF}}$) coincides with the previously defined extension $⨍_{\mathbb{R}^{dk}}^{\mathcal{R},\mathrm{ren}}$ (resp. $\sum_{\mathbb{N}^k}^{\mathcal{R},\mathrm{ren}}$). □

5 Meromorphic discrete sums of symbols on cones

5.1 Convex cones and matrices. A closed resp. open *convex cone* in $\mathbb{R}^k$ is a convex set

$$C := \sum_{j=1}^{J} \mathbb{R}_{\geq 0} v_j \quad \text{resp.} \quad C := \sum_{j=1}^{J} \mathbb{R}_{+} v_j$$

generated by non-zero vectors v_j, $j = 1, \ldots, J$, of $\mathbb{R}^k$. The dimension of the cone is the dimension of the linear space generated by the vectors $v_1, \ldots, v_J$. In the following, we focus on closed convex cones; similar results hold for open convex cones.

A convex cone together with the set of its generators is denoted by $C = \langle v_1 \ldots, v_J \rangle$. To a matrix $A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq J}$ with k rows and J columns we associate a convex cone

$$\mathcal{C}_A := \langle \textstyle\sum_{i=1}^{J} a_{i1} e_i, \ldots, \sum_{i=1}^{J} a_{iJ} e_i \rangle$$

so that for any $\vec{x} \in \mathbb{R}^k$ we have $\vec{x} \in \mathcal{C}_A$ if and only if there exists $\vec{y} \in \mathbb{R}_{\geq 0}^J$ such that $\vec{x} = A\vec{y}$. Alternatively, for $\vec{x} \in \mathbb{R}^k$ we have

$$\vec{x} \in \mathcal{C}_A \iff \text{there exist } \vec{n} \in \mathbb{Z}_{\geq 0}^J \text{ and } \vec{y} \in \Pi(v_1, \ldots, v_J) \text{ such that } \vec{x} = \vec{y} + A\vec{n}, \tag{24}$$

where (for a closed cone[12]) $\Pi(v_1, \dots, v_J) = \sum_{j=1}^{J} [0, 1[\, v_j$ is the semi-open parallelepiped generated by the v_j's.

Remark 5.1. The correspondence between matrices and cones is one-to-one up to permutations on the generators; whereas a permutation of the column vectors v_j modifies the matrix, it leaves the cone unchanged.

A convex cone in $\mathbb{R}^k$ is *pointed* if it does not contain a straight line. We consider pointed cones $\mathcal{C}_A$ associated with matrices A with non-negative coefficients. Since the generators v_j are non-zero, the matrix has no column of zeroes; each column contains at least one positive coefficient.

A convex cone $\langle v_1, \dots, v_J \rangle$ is called a *rational convex cone* if it is generated by vectors $v_1, \dots, v_J \in \mathbb{Q}^k$. A cone $\mathcal{C}_A$ associated with a matrix A is rational whenever the matrix $A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq J}$ has rational coefficients. As in [4] we simply call a rational convex cone a cone.

A cone $\langle v_1, \dots, v_J \rangle$ is *simplicial* if it is generated by independent vectors $v_1, \dots, v_J$ of $\mathbb{Q}^k$. A cone $\mathcal{C}_A$ associated with a matrix A is simplicial whenever the matrix $A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq J}$ has rank J. For a simplicial cone, the couple $(\vec{y}, \vec{n}) \in \Pi(v_1, \dots, v_J) \times \mathbb{Z}^J$ in (24) is unique.

A simplicial cone $\langle v_1, \dots, v_J \rangle$ is called is called *integral* if the generators $v_1, \dots, v_J$ lie in $\mathbb{Z}^k$ or equivalently, if the matrix A has integer coefficients. For a simplicial integral cone $\mathcal{C}_A$ we have

$$\vec{x} \in \mathcal{C}_A \cap \mathbb{Z}^k \iff \begin{array}{l} \text{there exist } \vec{n} \in \mathbb{Z}^J_{\geq 0} \text{ and } \vec{y} \in \Pi(v_1, \dots, v_J) \cap \mathbb{Z}^k \\ \text{such that } \vec{x} = \vec{y} + A\vec{n}. \end{array}$$

Let us recall that any cone can be subdivided into simplicial integral cones.

An integral cone is *unimodular* if $\Pi(v_1, \dots, v_J)$ has volume 1.

Example 5.2. The closed *Chen cone* of dimension k, which is the k-dimensional closed convex cone $\mathcal{C}_{\mathrm{Chen}}$ associated with the upper triangular matrix with coefficients $a_{ij} = 1$ if $i \leq j$ and $a_{ij} = 0$ if $i > j$ is an unimodular integral simplicial cone and for $f \in L^1(\mathbb{R}^k)$ we have

$$\sum_{\vec{x} \in \mathbb{Z}^k \cap \mathcal{C}_{\mathrm{Chen}}} f(\vec{x}) = \sum_{0 \leq n_k \leq \cdots < \leq n_1} f(n_1, \dots, n_k) = \sum_{\vec{n} \in \mathbb{Z}^k_{\geq 0}} f \circ A(\vec{n}).$$

More generally, multiple sums of the type $\sum_{\vec{n} \in \mathbb{Z}^k_{\geq 0}} f \circ A(\vec{n})$, where A is an integer matrix relate to sums on cones via the straightforward formula

$$\sum_{\vec{n} \in \mathbb{Z}^k_{\geq 0}} f \circ A(\vec{n}) = \sum_{\vec{x} \in \mathbb{Z}^k \cap \mathcal{C}_A} f(\vec{x}) \operatorname{card}(P_A(\vec{x}) \cap \mathbb{Z}^J_{\geq 0}),$$

---

[12] A similar description holds for an open cone setting instead $\Pi(v_1, \dots, v_J) = \sum_{j=1}^{J}]0, 1]\, v_j$.

with $f \in L^1(\mathbb{R}^k)$ and where we have set $P_A(\vec{x}) = \{\vec{y} \in \mathbb{R}^J \mid A\vec{y} = \vec{x}\}$. The number $\mathrm{card}(P_A(\vec{x}) \cap \mathbb{N}^J)$ of integral points in $P_A(\vec{x})$ has lead to many fascinating investigations in relation to the Euler–Maclaurin formula for polytopes on the one hand and to splines on the other hand, all of which lie outside of the scope of this article, but which we hope to get back to in forthcoming work.

5.2 Meromorphic extensions of discrete sums on cones. The following lemma generalises the elementary identity

$$\sum_{n=1}^{\infty} e^{-\varepsilon n} = \frac{1}{1-e^{-\varepsilon}} \quad \text{for all } \varepsilon > 0$$

used in (14) to similar sums on cones.

Lemma 5.3 (see Lemma 11 [4] and references therein). *Given a cone* $\mathcal{C} \subset \mathbb{R}^k_{\geq 0}$ *with non-zero generators* $v_1, \ldots, v_J$, *the map*

$$S_{\mathcal{C}}\colon \mathbb{R}^k_+ \to \mathbb{C}, \quad \vec{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_k) \mapsto \sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}^k} e^{-\langle \vec{\varepsilon}, \vec{x} \rangle}$$

is meromorphic with poles on hyperplanes determined by the edges of the cone. More precisely,

$$S_{\mathcal{C}}(\vec{\varepsilon}) = \frac{h(\vec{\varepsilon})}{\prod_{j=1}^{J} \langle v_j, \vec{\varepsilon} \rangle} \tag{25}$$

for some entire map h.

Proof. Using equation (24) we write elements of the cone $\mathcal{C}$ as

$$\vec{x} = \vec{y} + \sum_{j=1}^{k} n_j v_j, \quad n_j \in \mathbb{Z}_{\geq 0},$$

where $\vec{y} \in \Pi(v_1, \ldots, v_J)$. Hence we have

$$\begin{aligned} S_{\mathcal{C}}(\vec{\varepsilon}) = \sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}^k} e^{-\langle \vec{\varepsilon}, \vec{x} \rangle} &= \Big(\sum_{\vec{x} \in \Pi(v_1, \ldots, v_J) \cap \mathbb{Z}^k} e^{-\langle \vec{\varepsilon}, \vec{x} \rangle} \Big) \cdot \textstyle\prod_{j=1}^{J} \sum_{n_j=0}^{\infty} e^{-n_j \langle \vec{\varepsilon}, v_j \rangle} \\ &= \Big(\sum_{\vec{x} \in \Pi(v_1, \ldots, v_J) \cap \mathbb{Z}^k} e^{-\langle \vec{\varepsilon}, \vec{x} \rangle} \Big) \cdot \textstyle\prod_{j=1}^{J} \frac{1}{1-e^{-\langle \vec{\varepsilon}, v_j \rangle}}. \end{aligned}$$

The denominator does not vanish since $\langle \vec{\varepsilon}, v_j \rangle = \sum_{i=1}^{k} \varepsilon_i a_{ij} > 0$ for all $j = 1, \ldots, J$; indeed, the matrix (a_{ij}) has non-negative coefficients since $\mathcal{C} \subset \mathbb{R}^k_+$, and the vectors v_j being non-zero, for any $j \in \{1, \ldots, J\}$ there is some index i such that $a_{ij} > 0$. The poles around $\vec{\varepsilon} = 0$ are given by $\langle \vec{\varepsilon}, v_j \rangle$, $j = 1, \ldots, J$. □

The following proposition generalises (14) to cones.

Proposition 5.4. *Given a cone $\mathcal{C}_A \subset \mathbb{R}^k_{\geq 0}$ associated with a matrix A, the map*

$$(a_1, \ldots, a_k) \mapsto \sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{N}^k} x_1^{-a_1} \ldots x_k^{-a_k}$$

is holomorphic on the intersection of half-planes $\mathrm{Re}(a_{\tau(1)} + \cdots + a_{\tau(i)}) > s_i$, $i = 1, \ldots, k$, $\tau \in \Sigma_k$, *and extends to a meromorphic map, which to the k-tuple* $(a_1, \ldots, a_k)$ *assigns*

$$\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{N}^k} x_1^{-a_1} \ldots x_k^{-a_k} = \frac{1}{\Gamma(a_1) \ldots \Gamma(a_k)} \cdot \sum_{\tau \in \Sigma_k} \frac{h_{\tau, \vec{m}}(a_1, \ldots, a_k)}{\prod_{i=1}^k [(a_{\tau(1)} + \cdots + a_{\tau(i)} - s_i) \ldots (a_{\tau(1)} + \cdots + a_{\tau(i)} - s_i + m_i)]}$$

with simple poles $(a_1, \ldots, a_k) \in \mathbb{C}^k$ *on a countable set of affine hyperplanes*

$$a_{\tau(1)} + \cdots + a_{\tau(i)} \in \mathbb{Z}_{\leq 0} + s_i, \quad i = 1, \ldots, k, \ \tau \in \Sigma_k.$$

Here $h_{\tau, \vec{m}}$, $\tau \in \Sigma_I$, $\vec{m} \in \mathbb{Z}^k_{\geq 0}$, *is a holomorphic map on the domain*

$$H_{\tau, \vec{m}} := \bigcap_{i=1}^k \{\mathrm{Re}(a_{(\tau(1)} + \cdots + a_{\tau(i)}) + m_i > s_i\}.$$

For any index $i \in \{1, \ldots, k\}$, *the integer* s_i *is the number of columns of the matrix with "length" no larger than* i, *by which we mean that the number of columns indexed by* j *for which the smallest row index* i_j *beyond which the column under consideration has only zero coefficients is such that* $i_j \leq i$.

Proof. The proof closely follows that of [37]. For $\mathrm{Re}(a_i)$ sufficiently large, using an iterated Mellin transform we write (compare with (14))

$$\begin{aligned}
\sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{N}^k} x_1^{-a_1} \ldots x_k^{-a_k}
&= \sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{Z}^k} (x_1 + 1)^{-a_1} \ldots (x_k + 1)^{-a_k} \\
&= \frac{1}{\Gamma(a_1) \ldots \Gamma(a_k)} \int_0^\infty d\varepsilon_1 \varepsilon_1^{a_1 - 1} \ldots \int_0^\infty d\varepsilon_k \varepsilon_k^{a_k - 1} \sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{Z}^k} e^{-\sum_{i=1}^k \varepsilon_i \cdot x_i - \sum_{i=1}^k \varepsilon_i} \\
&= \frac{1}{\Gamma(a_1) \ldots \Gamma(a_k)} \int_0^\infty d\varepsilon_1 \varepsilon_1^{a_1 - 1} \ldots \int_0^\infty d\varepsilon_k \varepsilon_k^{a_k - 1} e^{-\sum_{i=1}^k \varepsilon_i} \sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}^k} e^{-\langle \vec{\varepsilon}, \vec{x} \rangle} \\
&= \frac{1}{\Gamma(a_1) \ldots \Gamma(a_k)} \int_0^\infty d\varepsilon_1 \ldots \int_0^\infty d\varepsilon_k \prod_{i=1}^k \varepsilon_i^{a_i - 1} \prod_{i=1}^J \langle v_j, \vec{\varepsilon} \rangle^{-1} h(\vec{\varepsilon}) e^{-\sum_{i=1}^k \varepsilon_i}
\end{aligned} \tag{26}$$

for some entire map h with the notation of (25). Here we have set $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$ and $\vec{x} = (x_1, \dots, x_k)$.

Let us decompose the space $\mathbb{R}_+^k$ of parameters $(\varepsilon_1, \dots, \varepsilon_k)$ in regions D_τ defined by $\varepsilon_{\tau(1)} \leq \dots \leq \varepsilon_{\tau(k)}$ for permutations $\tau \in \Sigma_k$. This splits the integral

$$\int_0^\infty d\varepsilon_1 \dots \int_0^\infty d\varepsilon_k \prod_{i=1}^k \varepsilon_i^{a_i-1} \prod_{i=1}^J \langle v_j, \vec{\varepsilon}\rangle^{-1} h(\vec{\varepsilon}) e^{-\sum_{i=1}^k \varepsilon_i}$$

into a sum of integrals $\int \dots \int_{D_\tau} \prod_{i=1}^k \varepsilon_i^{a_i-1} \prod_{i=1}^J \langle v_j, \vec{\varepsilon}\rangle^{-1} h(\vec{\varepsilon}) e^{-\sum_{i=1}^k \varepsilon_i} d\varepsilon_1 \dots d\varepsilon_k$.

Let us focus on the integral over the domain D given by $\varepsilon_1 \leq \dots \leq \varepsilon_k$; the results can then be transposed to other domains applying a permutation $a_i \to a_{\tau(i)}$ on the a_i. Note that a permutation $\tau \in \Sigma_k$ on the parameters ε_i in the above integral amounts to a permutation τ on the parameters a_i as can be seen from the change of variable $\varepsilon_i' := \varepsilon_{\tau(i)}$ since $\prod_{i=1}^k \varepsilon_i^{a_i-1} = \prod_{i=1}^k \varepsilon_{\tau(i)}^{a_{\tau(i)-1}}$.

Setting $\varepsilon_i = t_k \dots t_i$ on this domain introduces new variables $\vec{t} = (t_1, \dots, t_k)$ which vary in the domain

$$\Delta := \prod_{i=1}^{k-1} [0, 1] \times [0, \infty).$$

Since $v_j = \sum_{i=1}^k a_{ij} e_i \neq 0$ for all $j \in \{1, \dots, J\}$, for any $j \in \{1, \dots, J\}$, we can define $i_j \in \{1, \dots, k\}$ to be the largest index i such that $a_{ij} \neq 0$, which can intuitively be seen as the "length" of the j-th column since zeroes occupy the slots below the i_j-th coefficient of this column.

Performing the change of variable $(\varepsilon_1, \dots, \varepsilon_k) \mapsto (t_1, \dots, t_k)$ in the integral, which introduces a Jacobian determinant $\prod_{i=1}^k t_i^{i-1}$ we have

$$\langle v_j, \vec{\varepsilon}\rangle = \sum_{i=1}^k a_{ij} \varepsilon_i = t_k \dots t_{i_j} \Big(\sum_{i=1}^{i_j - 1} a_{ij} t_{i_j - 1} \dots t_1 + a_{i_j j} \Big).$$

Implementing this change of variable, we write the last sum arising in (26) as follows:

$$\begin{aligned} &\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{N}^k} x_1^{-a_1} \dots x_i^{-a_i} \\ &= \frac{1}{\Gamma(a_1) \dots \Gamma(a_k)} \\ &\quad \cdot \int_0^\infty dt_k \int_0^1 dt_1 \dots \int_0^1 dt_{k-1} \prod_{i=1}^k t_i^{i-1} \prod_{i=1}^k (t_k \dots t_i)^{a_i - 1} \prod_{j=1}^J (t_k \dots t_{i_j})^{-1} \tilde{h}(\vec{t}) \\ &= \frac{1}{\Gamma(a_1) \dots \Gamma(a_k)} \int_0^\infty dt_k \int_0^1 dt_1 \dots \int_0^1 dt_{k-1} \prod_{i=1}^k t_i^{a_1 + \dots + a_i - 1} \prod_{j=1}^J (t_k \dots t_{i_j})^{-1} \tilde{h}(\vec{t}) \end{aligned}$$

$$= \frac{1}{\Gamma(a_1)\dots\Gamma(a_k)} \int_\Delta \prod_{i=1}^k t_i^{a_1+\cdots+a_i-s_i-1} \tilde{h}(\vec{t}),$$

where we have set

$$\tilde{h}(\vec{t}) := h(t_k\dots t_1, t_k\dots t_2, \dots, t_k) \prod_{j=1}^J \Big(\sum_{i=1}^{i_j-1} a_{ij} t_{i_j-1}\dots t_1 + a_{i_j\, j}\Big)^{-1} e^{-\sum_{i=1}^k t_k\dots t_i}.$$

The non-negative integers

$$s_i = \mathrm{card}\{j \in \{1,\dots,J\},\ i \geq i_j\},$$

indexed by $i = 1,\dots,k$, are no larger than the number J of columns since they correspond to the number of columns whose length is no larger than i.

Integrating by parts with respect to each t_i, $i = 1,\dots,k$, introduces factors $\frac{1}{a_1+\cdots+a_i-s_i+m_i}$, $m_i \in \mathbb{Z}_{\geq 0}$, arising from taking primitives of $t_i^{a_1+\cdots+a_i-s_i-1}$ and differentiating $\tilde{h}(\vec{t})$. Note that $\tilde{h}$ is infinitely smoothing in the domain Δ and that the integral in t_k converges at infinity since the expression $h(t_k\dots t_1, t_k\dots t_2,\dots,t_k)$ involves exponentials $e^{-\sum_{i=1}^k t_k\dots t_i}$.

Summing the various integrals over the regions D_τ, with τ varying in Σ_k, amounts to summing over the domain D, the original integrals with a_i replaced by $a_{\tau(i)}$. This gives rise to a meromorphic extension $\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k} \prod_{i=1}^k x_i^{-a_i}$ on the whole complex plane such that the map $\Gamma(a_1)\dots\Gamma(a_k)\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k} \prod_{i=1}^k x_i^{-a_i}$ restricts on the domain

$$H := \bigcap_{\tau\in\Sigma_k} H_\tau \quad \text{with } H_\tau := \bigcap_{\tau\in\Sigma_k} \bigcap_{i=1}^k \{\mathrm{Re}(a_{\tau(1)}+\cdots+a_{\tau(i)}) > s_i\},$$

to a sum over permutations $\tau \in \Sigma_k$ of holomorphic expressions on H_τ given by

$$\frac{\int_\Delta \prod_{i=1}^k t_i^{a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i+m_i} \tilde{h}_\tau^{(m_1+\cdots+m_k)}(\vec{t})}{\prod_{i=1}^k((a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i)\dots(a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i+m_i))} + \text{bound. terms.}$$

The boundary terms on the domain Δ which arise from iterated m_i integrations by parts in each variable t_i, with the indices m_i chosen sufficiently large, are of the same type as the other terms since they are proportional to

$$\frac{\int_{\Delta'} \prod_{i=1}^k t_i^{a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i+m_i'} \tilde{h}_\tau^{(m_1'+\cdots+m_k')}(\vec{t})}{\prod_{i=1}^k((a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i)\dots(a_{\tau(1)}+\cdots+a_{\tau(i)}-s_i+m_i'))}$$

for some domain $\Delta' = \prod_{i=1}^{I'-1}[0,1]\times[0,\infty[$ for some $I' < I$ or $\Delta' = \prod_{i=1}^{I'-1}[0,1]$ for some $I' \leq I$ and some non-negative integers $m_i' \leq m_i$ with at least one $m_{i_0}' < m_{i_0}$.

To sum up, we get a meromorphic map $\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k}\prod_{i=1}^k x_i^{-a_i}$ such that on the domain H, the map $\Gamma(a_1)\dots\Gamma(a_k)\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k}\prod_{i=1}^k x_i^{-a_i}$ restricts to a holomorphic expression given by a sum over permutations $\tau\in\Sigma_k$ of expressions

$$\frac{h_{\tau,\vec{m}}(a_1,\dots,a_k)}{\prod_{i=1}^k((a_{\tau(1)}+\dots+a_{\tau(i)}-s_i)\dots(a_{\tau(1)}+\dots+a_{\tau(i)}-s_i+m_i))},$$

where $h_{\tau,\vec{m}}$ is a holomorphic map on the domain

$$H_{\tau,\vec{m}} := \bigcap_{i=1}^k \{\mathrm{Re}(a_{\tau(1)}+\dots+a_{\tau(i)})+m_i > s_i\}$$

indexed by non-negative integers m_i, $i=1,\dots,k$, and $\tau\in\Sigma_k$.

This iterated integration by parts procedure therefore produces a meromorphic map $\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k}\prod_{i=1}^k x_i^{-a_i}$ on the whole complex space $\mathbb{C}^k$ with simple poles on a countable set of affine hyperplanes

$$a_{\tau(1)}+\dots+a_{\tau(i)}\in s_i+\mathbb{Z}_{\le 0},\quad i\in\{1,\dots,k\},\ \tau\in\Sigma_k,$$

where the indices $s_i\le J$ are non-negative integers defined above, which only depend on the matrix via the indices j_i. □

Example 5.5. For Chen cones we have $J=k$ and $v_j=e_1+\dots+e_j$ so that, with the notations of the proof, $i_j=j$ and $s_i=i$ and the poles lie on hyperplanes $a_{\tau(1)}+\dots+a_{\tau(i)}\in\mathbb{Z}_{\le 0}+i$, $\tau\in\Sigma_k$. If $\mathrm{Re}(a_i)>1$ for any $i\in\{1,\dots,k\}$ then there is no hyperplane of poles passing through 0.

More precise results on the location of the poles [27] can be derived from direct inspection of these sums on Chen cones using an Euler–Maclaurin formula.

Proposition 5.4 also gives information on the poles of sums of polynomials on cones investigated in [4] from another point of view.

Corollary 5.6. *Given a cone $\mathcal{C}\subset\mathbb{R}^k_{\ge 0}$ associated with a matrix A and any polynomial $P(\vec{x})=\sum_{|\alpha|\le N}c_\alpha x_1^{\alpha_1}\dots x_k^{\alpha_k}$ the map*

$$(z_1,\dots,z_k)\mapsto\sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k}P(\vec{x})x_1^{-z_1}\dots x_k^{-z_k}$$

is meromorphic with hyperplanes of poles passing through zero given by $z_{\tau(1)}+\dots+z_{\tau(i)}=0$, $i=1,\dots,k$, $k\in\mathbb{N}$.

Proof. This result easily follows from the location of the poles described above with a_i replaced by $-\alpha_i+z_i$ for some non-negative coefficients α_i, which are therefore situated on hyperplanes

$$z_{\tau(1)}+\dots+z_{\tau(i)}=\alpha_{\tau(1)}+\dots+\alpha_{\tau(i)}+s_i\in\mathbb{Z}_{\le 0}\quad\text{for all } i\in\{1,\dots,k\}.$$

The hyperplanes of poles containing 0 are therefore

$$z_{\tau(1)} + \cdots + z_{\tau(i)} = 0, \quad i = 1, \dots, k, \ \tau \in \Sigma_k. \qquad \square$$

Remark 5.7. This also generalises results of [27] where it was shown that these maps are actually holomorphic at $\vec{z} = 0$ when the sum is taken over a Chen cone.

5.3 Discrete sums of holomorphic symbols with conical constraints. We consider a holomorphic regularisation procedure $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R})$ which sends a polyhomogeneous symbol $\sigma \in \mathrm{CS}(\mathbb{R})$ of order a to a holomorphic family $\sigma(z) \in \mathrm{CS}(\mathbb{R})$ of polyhomogeneous symbols of non-constant affine order $\alpha(z) = a - qz$ for some fixed positive q. The aim of this section is to prove the following theorem.

Theorem 5.8. *Let $\mathcal{C}_A$ be a cone in $\mathbb{R}^k_{\geq O}$ associated with a matrix $A = (a_{ij})$ and let $\sigma_1, \dots, \sigma_k \in \mathrm{CS}(\mathbb{R})$ be polyhomogeneous symbols on $\mathbb{R}$ of order a_i which are sent via $\mathcal{R}$ to $\sigma_i(z)$ of order $\alpha_i(z) = a_i - qz$.*

There are non-negative integers $s_{\tau(i)} \leq J$ with i varying in $\{1, \dots, k\}$ and τ varying in Σ_k such that the map

$$(z_1, \dots, z_k) \mapsto \sum_{\mathcal{C}_A \cap \mathbb{Z}^k} \bigotimes_{i=1}^k \sigma_i(z_i)$$

is holomorphic on the domain

$$H := \bigcap_{\tau \in \Sigma_k} \bigcap_{i=1}^{k} \{\mathrm{Re}(\alpha_{\tau(1)}(z_{\tau(1)}) + \cdots + \alpha_{\tau(i)}(z_{\tau(i)})) < -s_i\}$$

and extends to a meromorphic map on the complex plane with simple poles

$$z_{\tau(1)} + \cdots + z_{\tau(i)} \in \frac{a_{\tau(1)} + \cdots + a_{\tau(i)} + s_i + \mathbb{Z}_{\leq 0}}{q}, \quad i \in \{1, \dots, k\}, \ \tau \in \Sigma_k,$$

located on a countable set of affine hyperplanes. The hyperplanes of poles passing through the origin are of the type $z_{\tau(1)} + \cdots + z_{\tau(i)} = 0$. Hyperplanes of poles do not pass through the origin if all the partial sums $\alpha_{\tau(1)} + \cdots + \alpha_{\tau(i)}$, $i \in \{1, \dots, k\}$, with τ in Σ_k, of the orders of the symbols do not take integer values, in which case the map

$$\vec{z} \mapsto \sum_{\mathcal{C}_A \cap \mathbb{Z}^k} \otimes_{i=1}^k \sigma_i(z_i)$$

is holomorphic in a neighborhood of 0.

Proof. Let $\vec{\sigma} := \sigma_1 \otimes \cdots \otimes \sigma_k$ be a tensor product of polyhomogeneous symbols σ_i on $\mathbb{R}$ of order b_i, $i \in \{1, \dots, k\}$. Using (19) we write

$$\sigma_i(x) = \sum_{j_i=0}^{N_i-1} \sigma_{i,b_i-j_i}(x)\chi(x) + \sigma_i^{(N_i)}(x)\chi(x) = \sum_{j_i=0}^{N_i-1} c_{i,j_i} |x_i|^{b_i-j_i} \chi(x) + \sigma_i^{(N_i)}(x)\chi(x),$$

where N_i, $i = 1, \dots, I$, are positive integers, $\sigma_{i,b_i-j_i}(x) = c_{i,j_i}(x)|x|^{b_i-j_i}$ are positively homogeneous functions of degree $b_i - j_i$, with $c_{i,i_j}(x) = \sigma_{i,b_i-j_i}(x/|x|)$, $i = 1, \dots, k$, and $\sigma_i^{(N_i)}$, $i = 1, \dots, k$, polyhomogeneous symbols of order b_i with real part no larger than $\mathrm{Re}(b_i) - N_i$. Note that $c_{i,j_i} := c_{i,i_j}(x) = c_{i,i_j}(1)$ is constant for non-negative x. Here χ is a smooth cut-off function which vanishes around zero and is identically one outside the unit ball and zero in a neighborhood of zero.

In view of (18) we have

$$\prod_{i=1}^{k} \sigma_i(x_i) = \lim_{N\to\infty} \prod_{i=1}^{k} \sum_{j_i=0}^{N-1} \sigma_{i,b_i-j_i}(x_i)\chi(x_i) = \lim_{N\to\infty} \prod_{i=1}^{k} \sum_{j_i=0}^{N-1} c^i_{j_i}(x_i)|x_i|^{b_i-j_i}\chi(x_i). \tag{27}$$

in the Fréchet topology on classical symbols of constant order. By Proposition 5.4, for a fixed integer N the map

$$\begin{aligned}(b_1, \dots, b_k) \mapsto \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} \sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{Z}^k} \prod_{i=1}^{k} \sigma_{i,b_i-j_i}(x_i)\chi(x_i) \\ = \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} c^1_{j_1} \dots c^k_{j_k} \sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k} \prod_{i=1}^{k} x_i^{b_i-j_i}\end{aligned}$$

is holomorphic on the intersection of half-planes $\mathrm{Re}(b_{\tau(1)} + \cdots + b_{\tau(i)}) < -s_i$ with $i = 1, \dots, k$, $\tau \in \Sigma_k$, and extends to a meromorphic map

$$\begin{aligned}(b_1, \dots, b_k) \mapsto \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} \sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{Z}^k} \prod_{i=1}^{k} \sigma_{i,b_i-j_i}(x_i)\chi(x_i) \\ := \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} c^1_{j_1} \dots c^k_{j_k} \sum_{\vec{x}\in\mathcal{C}_A\cap\mathbb{N}^k} \prod_{i=1}^{k} x_i^{b_i-j_i}\end{aligned}$$

with poles $(b_1, \dots, b_k) \in \mathbb{C}^k$ on a countable set of affine hyperplanes

$$b_{\tau(1)} + \cdots + b_{\tau(i)} + s_i \in \mathbb{Z}_{\geq 0}, \quad i = 1, \dots, k, \ \tau \in \Sigma_k.$$

Substituting $\alpha_i(z_i) = a_i - qz_i$ for b_i implies that for any positive integer N the map

$$(z_1, \dots, z_k) \mapsto \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} \sum_{\mathcal{C}_A\cap\mathbb{Z}^k} \prod_{i=1}^{k} \sigma_{i,\alpha_i(z_i)-j_i}(x_i)$$

is holomorphic on the intersection of half-planes

$$H := \bigcap_{k\in\mathbb{N}} \bigcap_{\tau\in\Sigma_k} (\alpha_{\tau(1)}(z_{\tau(1)}) + \cdots + \alpha_{\tau(i)}(b_{\tau(i)})) < -s_i$$

and extends to a meromorphic map

$$(z_1, \dots, z_k) \mapsto \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} \sum_{\mathcal{C}_A\cap\mathbb{Z}^k} \prod_{i=1}^{k} \sigma_{i,\alpha_i(z_i)-j_i}(x_i)$$

on the whole complex plane with poles

$$z_{\tau(1)} + \cdots + z_{\tau(k)} \in \frac{\alpha_{\tau(1)} + \cdots + \alpha_{\tau(i)} + s_i + \mathbb{Z}_{\leq 0}}{q}, \quad i \in \{1, \dots, k\},\ \tau \in \Sigma_k,$$

located on a countable set of affine hyperplanes. Since the domain of holomorphicity and the pole structure are independent of N, taking the limit as $N \to \infty$ as in (27) we deduce that the map

$$\vec{z} \mapsto \sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{Z}^k} \prod_{i=1}^{k} \sigma_i(z_i)(x_i)$$

is also holomorphic on H and extends to a meromorphic map on the complex plane

$$\vec{z} \mapsto \sum_{\vec{x} \in \mathcal{C}_A \cap \mathbb{Z}^k} \prod_{i=1}^{k} \sigma_i(z_i)(x_i)$$

with the same pole structure. □

Theorem 5.8 extends to any cone $\mathcal{C} \subset \mathbb{R}^k$ by additivity on sets of disjoint cones.

We first observe that to a cone $C = \langle v_1, \dots, v_J \rangle$ in $\mathbb{R}^k_{\geq 0}$ corresponds a unique meromorphic expression $\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{N}^k} \prod_{i=1}^{k} \sigma_i(z_i)(x_i)$; indeed, matrices A which give rise to the same cone $\mathcal{C} = \mathcal{C}_A$ differ by a permutation of their column vectors, which does not affect the sum as can be seen by analytic continuation from the fact that it does not affect the sum whenever it converges.

Now a general convex cone $\mathcal{C} \subset \mathbb{R}^k$ can be written as a finite union of cones $\mathcal{C} = \bigcup_{j=1}^{J} \mathcal{C}_j$ with each cone $\mathcal{C}_j \subset \mathbb{R}^k$ contained in the intersection $\bigcap_{i=1}^{k} S_i^j$ of semi-hyperplanes $S_i^j : \varepsilon_i^j x_i \leq 0$, where ε_i^j is either plus or minus one. The map $\varepsilon^j : (x_1, \dots, x_k) \mapsto (\varepsilon_1^j x_1, \dots, \varepsilon_k^j x_k)$ sends $\mathcal{C}^j$ to a cone in $\mathcal{C}_j' \subset \mathbb{R}^k_{\geq 0}$ so that replacing the symbols σ_i by $\sigma_i' : x \mapsto \sigma_i(\varepsilon_i^j x)$ amounts to summing $\sigma_1' \otimes \cdots \otimes \sigma_k'$ over $\mathcal{C}_j'$ instead of summing $\sigma_1 \otimes \cdots \otimes \sigma_k$ over $\mathcal{C}_j$. Since the new symbols σ_i' have the same order as the original symbols, the pole structure is unaffected by this transformation.

Corollary 5.9. *Let $\mathcal{C} = \langle v_1, \dots, v_J \rangle$ be a cone in $\mathbb{R}^k$ generated by J non-zero vectors $v_1, \dots, v_J$ and let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on* $\mathrm{CS}(\mathbb{R})$. *For polyhomogeneous symbols $\sigma_1, \dots, \sigma_k \in \mathrm{CS}(\mathbb{R})$ on $\mathbb{R}$ of order a_i which are sent via $\mathcal{R}$ to $\sigma_i(z)$ of holomorphic order $\alpha_i(z) = \alpha_i - qz$, the map*

$$(z_1, \dots, z_k) \mapsto \sum_{\mathcal{C} \cap \mathbb{Z}^k} \bigotimes_{i=1}^{k} \sigma_i(z_i)$$

is holomorphic on the intersection of the domains

$$\{\mathrm{Re}(\alpha_{\tau(1)}(z_{\tau(1)}) + \cdots + \alpha_{\tau(i)}(z_{\tau(i)})) < -J\}$$

and extends to a meromorphic map on the complex plane with simple poles

$$a_{\tau(1)} + \cdots + a_{\tau(i)} \in \mathbb{Z}_{\leq 0} + s_i, \quad i = 1, \dots, k,\ \tau \in \Sigma_k,$$

located on a countable set of affine hyperplanes which do not pass through the origin if all the partial sums of the orders $\alpha_{\tau(1)} + \cdots + \alpha_{\tau(i)}$, $i \in \{1, \dots, k\}$, $\tau \in \Sigma_k$, of the symbols do not take integer values.

Let us introduce the set

$$\mathcal{A}_k^{\mathrm{cones}} = \{(\sigma_1 \otimes \cdots \otimes \sigma_k; A),\ \sigma_i \in \mathrm{CS}(\mathbb{R}),\ \text{with } A \text{ a matrix of size } k \times k \text{ with non-zero columns}\}.$$

The Whitney sum of matrices extends to a product

$$(\vec{\sigma}; A) \bullet (\vec{\sigma}'; A') := (\vec{\sigma} \otimes \vec{\sigma}'; A \oplus A') \tag{28}$$

so that $\mathcal{A}^{\mathrm{cones}} := \bigcup_{k=0}^{\infty} \mathcal{A}_k^{\mathrm{cones}}$ equipped with $\bullet$ is a (non-commutative) filtered algebra.

The above results can be reformulated as follows.

Corollary 5.10. *Given a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto \sigma(z)$, the map*

$$\Phi^{\mathcal{R}}\colon \mathcal{A}^{\mathrm{cones}} \to \mathcal{B}, \quad (\vec{\sigma}, A) \mapsto (\vec{z} \mapsto \textstyle\sum_{\mathcal{C}_A \cap \mathbb{Z}^k} \vec{\sigma}(\vec{z})), \tag{29}$$

where $\mathcal{C}_A \subset \mathbb{R}^k$ is the cone associated with the matrix A, is a morphism of filtered algebras. Here $\mathcal{B}$ is the filtered algebra of meromorphic functions described in (1.7).

6 Multiple integrals of symbols with linear constraints

We consider multiple integrals with linear constraints, which we describe in terms of matrices as we did for the conical constraints arising in discrete sums studied in the previous section.

6.1 Linear constraints in terms of matrices. To a matrix B with real coefficients

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1L} \\ \dots & \dots & \dots & \dots \\ b_{I1} & b_{I2} & \dots & b_{IL} \end{pmatrix}$$

and symbols $\sigma_i \in \mathrm{CS}(\mathbb{R}^d), i = 1, \dots, I$ we associate the map

$$(\xi_1, \dots, \xi_L) \mapsto (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \dots, \xi_L) := \sigma_1\Big(\sum_{l=1}^{L} b_{1l}\xi_l\Big) \dots \sigma_I\Big(\sum_{l=1}^{L} b_{Il}\xi_l\Big).$$

We want to investigate the corresponding multiple integral with linear constraints:

$$\int_{\mathbb{R}^{nL}} (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \dots, \xi_L)\, d\xi_1 \dots d\xi_L.$$

Feynman diagrams without external momenta for bosonic theories with polynomial interactions typically give rise to (possibly divergent) integrals of the type

$$\int_{\mathbb{R}^{nL}} (\sigma^{\otimes I} \circ B)(\xi_1, \ldots, \xi_L)\, d\xi_1 \ldots d\xi_L,$$

with $\sigma(\xi) = \frac{1}{(m^2+|\xi_1|^2)}$. Allowing for external momenta would lead to affine constraints, a case which lies outside the scope of this article but which we hope to investigate in forthcoming work. Constraints on the momenta follow from the conservation of momentum as it flows through the diagram and L corresponds to the number of loops in the diagram.

Remark 6.1. For fixed σ, the correspondence between Feynman type integrals without external momenta and matrices is not one-to-one. Indeed,

(1) a permutation τ in Σ_I on the rows of the matrix amounts to relabelling the symbols σ_i in the tensor product, which does not affect their product when $\sigma_i = \sigma$ is independent of i;

(2) a permutation τ in Σ_L on the columns of the matrix amounts to relabelling the variables ξ_l, which does not affect the Feynman integral provided the Fubini property holds.

6.2 Holomorphicity. We consider a holomorphic regularisation procedure $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R}^d)$ which sends a polyhomogeneous symbol $\sigma \in \mathrm{CS}(\mathbb{R}^d)$ of order a to a holomorphic family $\sigma(z) \in \mathrm{CS}(\mathbb{R}^d)$ of polyhomogeneous symbols of non-constant affine order $\alpha(z) = a - qz$.

A tensor product $\vec{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_I \in \mathcal{T}(\mathrm{CS}(\mathbb{R}^d))$ is sent to

$$\vec{\sigma}(\vec{z}) := \widetilde{\mathcal{R}}(\sigma)(\vec{z}) := \sigma_1(z_1) \otimes \cdots \otimes \sigma_I(z_I),$$

where we have set $\vec{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$.

Proposition 6.2. *Let $\sigma_i \in \mathrm{CS}(\mathbb{R}^d)$, $i = 1 \ldots, I$ of order a_i. Given a matrix $B = (b_{il})$ of size $I \times L$ and rank L, the map*

$$\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \vec{\sigma}(\vec{z}) \circ B$$

is holomorphic on the domain $D = \left\{\vec{z} \in \mathbb{C}^I,\ \mathrm{Re}(z_i) > \frac{a_i+d}{q} \text{ for all } i \in \{1, \ldots, I\}\right\}$.

Proof. The symbol property of each σ_i yields the existence of a constant C such that

$$|\vec{\sigma}(\vec{z}) \circ B(\xi_1, \ldots, \xi_L)| \leq C \prod_{i=1}^{I} \langle \sum_{l=1}^{L} b_{il}\xi_l \rangle^{\mathrm{Re}(\alpha_i(z_i))} \leq C \prod_{i=1}^{I} \langle \sum_{l=1}^{L} b_{il}\xi_l \rangle^{-q\mathrm{Re}(z_i)+a_i},$$

where we have set $\langle \eta \rangle := \sqrt{1 + |\eta|^2}$.

Thus, for $\mathrm{Re}(z_i) \geq \beta_i > 0$ we have

$$|\tilde{\sigma}(\vec{z}) \circ B(\xi_1, \ldots, \xi_L)| \leq C \prod_{i=1}^{I} \langle \sum_{l=1}^{L} b_{il}\xi_l \rangle^{-q\,\beta_i + \alpha_i}.$$

We claim that the map $(\xi_1, \ldots, \xi_L) \mapsto \langle \sum_{l=1}^{L} b_{il}\xi_l \rangle^{-q\,\beta_i + \alpha_i}$ lies in $L^1(R^{nL})$ if $\beta_i > \frac{\alpha_i + d}{q}$. Indeed, the matrix B being of rank L by assumption, we can extract an invertible $L \times L$ matrix D. Assuming for simplicity (and without loss of generality, since this assumption holds up to permutation of the rows and columns) that it corresponds to the L first rows of B, we write

$$\prod_{i=1}^{I} \langle \sum_{l=1}^{L} b_{il}\xi_l \rangle^{-q\,\beta_i + a_i} = \prod_{i=1}^{I} \rho_i \circ B(\xi_1, \ldots, \xi_L) \leq \prod_{i=1}^{L} \rho_i \circ D(\xi_1, \ldots, \xi_L),$$

where we have set $\rho_i(\eta) := \langle \eta \rangle^{-q\beta_i + a_i}$ and used the fact that $\rho_i(\eta) \geq 1$ and $-q\beta_i + a_i < -d$.

But

$$\int_{\mathbb{R}^{nL}} \bigotimes_{i=1}^{L} \rho_i \circ D = |\det D| \prod_{i=1}^{L} \int_{\mathbb{R}^d} \rho_i$$

converges as a product of integrals of symbols of order $< -d$ so that, by dominated convergence, $\widetilde{\mathcal{R}}(\vec{\sigma})(\vec{z}) \circ B$ lies in $L^1(\mathbb{R}^{dL})$ for any complex number $\vec{z} \in D$.

On the other hand, the derivatives in z of holomorphic symbols are of the same order as the original symbols (see e.g. [32]), the differentiation possibly introducing logarithmic terms. Replacing $\sigma_1(z_1), \ldots, \sigma_I(z_I)$ by $\partial_{z_1}^{\gamma_1}\sigma_1(z_1), \ldots, \partial_{z_I}^{\gamma_I}\sigma_I(z_I)$ in the above inequalities, by a similar procedure we show that for $\mathrm{Re}(z_i) \geq \beta_i > \frac{a_i + d}{q}$ the map $\vec{z} \mapsto \vec{\sigma}(\vec{z}) \circ B$ is uniformly bounded by an L^1 function. The holomorphicity of $\vec{z} \mapsto \int_{\mathbb{R}^{dL}} \vec{\sigma}(\vec{z}) \circ B$ then follows. □

6.3 Meromorphic extensions of multiple integrals with linear constraints. With the above notations, we now show the existence of meromorphic extensions for integrals $\vec{z} \mapsto \int_{\mathbb{R}^{dL}} \vec{\sigma}(\vec{z}) \circ B$.

Let us recall on an example how such integrals are handled in the physics literature. A common approach used by physicists is the Mellin transform method, which involves the so-called Schwinger parameters:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t}\, dt \quad \text{for all } \lambda > 0.$$

An alternative approach is to use so-called Feynman parameters via the following observation:

$$\prod_{i=1}^{k} \lambda_i^{-s_i} = \frac{\Gamma(\sum_{i=1}^{k} s_i)}{\prod_{i=1}^{k} \Gamma(s_i)} \int_0^1 \left(\prod_{i=1}^{k} dt_i\, t_i^{s_i - 1} \right) \frac{\delta(1 - \sum_{i=1}^{n} t_i)}{(\sum_{i=1}^{k} t_i \lambda_i)^s}.$$

In what follows we use the Schwinger parameter method to analyse the poles. As a warm-up, let us illustrate this approach on the following example.

Example 6.3. Writing

$$(1+|k_i|^2)^{-s_i} = \frac{1}{\Gamma(s_i)} \int_0^\infty t_i^{s_i-1} e^{-|k_i|^2 t_i}\, dt_i,$$

and using the fact that $(1+|k_1|^2)t_1 + (1+|k_2|^2)t_2 + (1+|k_1+k_2|^2)t_3 = (t_1 + t_3)|k_1 + \frac{t_3}{t_1+t_3}k_2|^2 + |k_2|^2(-\frac{t_3^2}{t_1+t_3} + t_2 + t_3) + t_1 + t_2 + t_3$, the Schwinger method yields for example, (2):

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|k_1|^2)^{-s_1}(1+|k_2|^2)^{-s_2}(1+(k_1+k_2)^2)^{-s_3}\, dk_1\, dk_2 \\
&\quad = \frac{1}{\prod_{i=1}^3 \Gamma(s_i)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\int_{(\mathbb{R}_+)^3} dt_1\, dt_2\, dt_3\, t_1^{s_1-1} t_2^{s_2-1} t_3^{s_3-1} \\
&\qquad \cdot e^{-(1+|k_1|^2)t_1 + (1+|k_2|^2)t_2 + (1+(k_1+k_2)^2)t_3} \Big) dk_1\, dk_2 \\
&\quad = \frac{1}{\prod_{i=1}^3 \Gamma(s_i)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3\, t_1^{s_1-1} t_2^{s_2-1} t_3^{s_3-1} \Big) \\
&\qquad \cdot e^{-(t_1+t_3)(k_1 - \frac{t_3}{t_1+t_3}k_2)^2 + |k_2|^2(-\frac{t_3^2}{t_1+t_3} + t_1 + t_2) - (t_1+t_2+t_3)}\, dk_1\, dk_2 \\
&\quad = \frac{1}{\prod_{i=1}^3 \Gamma(s_i)} \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 t_1^{s_1-1} t_2^{s_2-1} t_3^{s_3-1} e^{-(t_1+t_2+t_3)} \\
&\qquad \cdot \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-(t_1+t_3)|k_1|^2 - |k_2|^2(-\frac{t_3^2}{t_1+t_3} + t_1 + t_2)}\, dk_1\, dk_2 \\
&\quad = \frac{1}{\prod_{i=1}^3 \Gamma(s_i)} \frac{4\pi^n}{\Gamma(\frac{n}{2})^2} \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3\, t_1^{s_1-1} t_2^{s_2-1} t_3^{s_3-1} e^{-(t_1+t_2+t_3)} \\
&\qquad \cdot \int_0^\infty \int_0^\infty e^{-(t_1+t_3)r_1^2 - (-\frac{t_3^2}{t_1+t_3} + t_1 + t_2)r_2^2}\, r_1^{n-1}\, r_2^{n-1}\, dr_1\, dr_2 \\
&\quad = \frac{\pi^n}{\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)} \int_{(\mathbb{R}+)^3} dt_1\, dt_2\, dt_3\, t_1^{s_1-1} t_2^{s_2-1} t_3^{s_3-1} e^{-(t_1+t_2+t_3)} \\
&\qquad \cdot (t_1+t_2)^{-\frac{n}{2}} (-\frac{t_3^2}{t_1+t_3} + t_1 + t_2)^{-\frac{n}{2}}.
\end{aligned}$$

Note that singularities at infinity are taken care of by the exponential term $e^{-(t_1+t_2+t_3)}$ and integration by parts shows how singularities at zero are controlled by the values of the s_i.

Theorem 6.4. *Let $\xi \mapsto \sigma_i(\xi) := \tau_i(|\xi|)$, $\tau_i \in \mathrm{CS}(\mathbb{R}_+)$, $i = 1, \dots, I$ be* ***radial***[13] *polyhomogeneous symbols of order a_i which are sent via $\mathcal{R}$ to $\xi \mapsto \sigma_i(z)(\xi) := \tau_i(z)(|\xi|)$ of non-constant affine order $\alpha_i(z) = -qz_i + a_i$.*

[13] By radial we mean that it is of the form $\tau(|x|)$ for some symbol τ on $\mathbb{R}$.

For any matrix B *of size* $I \times L$ *and rank* L*, the map*

$$\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \tilde{\vec{\sigma}}(\vec{z}) \circ B,$$

which is well defined and holomorphic on the domain $D = \{\vec{z} \in \mathbb{C}^I, \operatorname{Re}(z_i) > \frac{a_i+d}{q}$ *for all* $i \in \{1, \dots, I\}\}$*, extends to a meromorphic map on the whole complex plane with poles*

$$z_{\tau(1)} + \cdots + z_{\tau(i)} \in \frac{\alpha_{\tau(1)} + \cdots + \alpha_{\tau(i)} + d\, s_i + \mathbb{Z}_{\leq 0}}{q}, \quad i \in \{1, \dots, I\},\ \tau \in \Sigma_I,$$

located on a countable set of affine hyperplanes. The integer s_i *is the number of columns of the matrix with "length" no larger than* i *by which we mean the number of columns indexed by* j *for which the smallest line index* i_j *beyond which the column under consideration has only zero coefficients is such that* $i_j \leq i$*.*

The hyperplanes of poles passing through the origin are of the type $z_{\tau(1)} + \cdots + z_{\tau(i)} = 0$*, with* $\tau \in \Sigma_k$*. Hyperplanes of poles do not pass through the origin if none of the partial sums* $a_{\tau(1)} + \cdots + a_{\tau(i)}$ *of the orders* a_i *are integers so that the map*

$$\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \tilde{\vec{\sigma}}(\vec{z}) \circ B$$

is holomorphic in a neighborhood of 0*.*

Proof. We only provide the main steps of the proof, which was carried out in detail in [30]. It is similar to that of Proposition 5.4 and closely follows Speer's proof [37], using iterated Mellin transforms and integrations by parts.

We proceed in several steps, first reducing the problem to step matrices B, then to symbols of the type $\sigma_i : \xi \mapsto (\xi^2 + 1)^{a_i}$ and finally proving the meromorphicity for such symbols and matrices.

Step 1: Reduction to step matrices.

We call an $I \times L$ matrix B with real coefficients a step matrix if it fulfills the following condition:

$$\text{there exist } i_1 < \cdots < i_L \text{ in } \{1, \dots, I\} \text{ such that } b_{il} = 0 \text{ if } i > i_l \text{ and } b_{i_l, l} \neq 0. \quad (30)$$

Lemma 6.5. *If Theorem 6.4 holds for step matrices then it holds for any matrix* B*.*

Step 2: Reduction to symbols $\sigma_i : \xi \mapsto \langle \xi \rangle^{\alpha_i}$.

Let us first recall the asymptotic behaviour of classical radial symbols. Given a radial polyhomogeneous symbol $\sigma : \xi \mapsto \tau(|\xi|)$ on $\mathbb{R}^d$ of order a, $\tau \in \mathrm{CS}(\mathbb{R}_+)$ of order a, there are real numbers c_j, $j \in \mathbb{Z}_{\geq 0}$ such that

$$\sigma(\xi) \sim \sum_{j=0}^{\infty} c_j \langle \xi \rangle^{a-j},$$

where $\sim$ stands for the equivalence of symbols modulo smoothing symbols and where we have set $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

Let $\xi \mapsto \sigma_1(\xi) := \tau_1(|\xi|), \ldots, \xi \mapsto \sigma_I(\xi) := \tau_I(|\xi|)$ be radial polyhomogeneous symbols on $\mathbb{R}^d$ of orders $a_1, \ldots, a_I$ respectively. Using (19) we write

$$\sigma_i(\xi) = \sum_{j_i=0}^{N_i-1} \tau_{i,a_i-j_i}(|\xi|) + \tau_i^{(N_i)}(|\xi|)\chi(|\xi|) = \sum_{j_i=0}^{N_i-1} c_{j_i}^i \langle\xi\rangle^{\alpha_i-j_i} + \tilde{\tau}_i^{(N_i)}(|\xi|),$$

where N_i, $i = 1, \ldots, I$, are positive integers, τ_{i,α_i-j_i}, $i = 1, \ldots, I$ are homogeneous functions of degree $\alpha_i - j_i$, $\tau_i^{(N_i)}, \tilde{\tau}_i^{(N_i)}$, $i = 1, \ldots, I$ polyhomogeneous symbols of order with real part no larger than $\mathrm{Re}(\alpha_i) - N_i$ and where we have set $c_{j_i}^i := \tau_{i,a_i-j_i}(1)$, $i = 1, \ldots, I$.

In view of (18) we have

$$\prod_{i=1}^{I} \sigma_i(\xi_i) = \lim_{N\to\infty} \sum_{j_1=0}^{N-1} \cdots \sum_{j_I=0}^{N-1} c_{j_1}^1 \ldots c_{j_I}^I \langle\xi_1\rangle^{\alpha_1-j_1} \ldots \langle\xi_I\rangle^{\alpha_I-j_I}$$

in the Fréchet topology on symbols of constant order.

Lemma 6.6. *If Theorem 6.4 holds for symbols $\sigma_i : \xi \mapsto \langle\xi\rangle^{\alpha_i}$ then it holds for all classical radial symbols.*

Step 3: The case of symbols $\sigma_i : \xi \mapsto \langle\xi\rangle^{\alpha_i}$ and step matrices.

We are therefore left to prove the statement of the theorem for an $I \times L$ matrix B with real coefficients which fulfills condition (30) and symbols $\sigma_i : \xi \mapsto \langle\xi\rangle^{\alpha_i}$. As previously observed, such a matrix has rank L.

The following result is proved in [30] along the lines of the proof of Proposition 5.4.

Proposition 6.7. *Let $B := (b_{il})_{i=1,\ldots,I;l=1,\ldots,L}$ be a matrix with property (30). The map*

$$(a_1, \ldots, a_I) \mapsto \int_{(\mathbb{R}^n)^L} \prod_{i=1}^{I} \Big\langle \sum_{l=1}^{L} b_{il}\xi_l \Big\rangle^{a_i} d\xi_1 \ldots d\xi_L,$$

which is holomorphic on the domain $D := \{(a_1, \ldots, a_I) \in \mathbb{C}^I, \mathrm{Re}(a_i) < -d$ for all $i \in \{1, \ldots, I\}\}$, has a meromorphic extension to the complex plane

$$(a_1, \ldots, a_I) \mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L} \prod_{i=1}^{I} \Big\langle \sum_{l=1}^{L} b_{il}\xi_L \Big\rangle^{a_i} d\xi_1 \ldots d\xi_L$$

$$:= \frac{1}{\prod_{i=1}^{I} \Gamma(-a_i/2)}$$

$$\cdot \sum_{\tau\in\Sigma_I} \frac{h_{\tau,\vec{m}}(a_1, \ldots, a_I)}{\prod_{i=1}^{I}[(a_{\tau(1)} + \cdots + a_{\tau(i)} + d\, s_i) \ldots (a_{\tau(1)} + \cdots + a_{\tau(i)} + d\, s_i - 2m_i)]}$$

for some holomorphic map $h_{\tau,\vec{m}}$ *on the domain*

$$\bigcap_{i=1}^{I}\{\mathrm{Re}(a_{\tau(1)}+\cdots+a_{\tau(i)})+2m_i<-d\,s_i\},$$

with τ *in* Σ_I *and* $\vec{m}:=(m_1,\dots,m_I)$ *a multi-index of non-negative integers. The integer* s_i *is the number of columns of the matrix with "length" no larger than* i *by which we mean the number of columns indexed by* j *for which the smallest row index* i_j *beyond which the column under consideration has only zero coefficients is such that* $i_j\leq i$.

The poles

$$a_{\tau(1)}+\cdots+a_{\tau(i)}\in -d\,s_i+\mathbb{Z}_{\geq 0}\quad \text{with } \tau\in\Sigma_I,\ i\in\{1,\dots,I\},$$

of this meromorphic extension lie on a countable set of affine hyperplanes.

This finishes the proof of the theorem. □

Let us consider the subset of $\mathcal{T}(\mathrm{CS}^{\mathrm{rad}}(\mathbb{R}^n))_{\mathrm{lin}}$ (using the notations of the Incipit) defined by $\mathcal{A}:=\bigcup_{I=1}^{\infty}\mathcal{A}_I^{\mathrm{lin}}$ with

$$\mathcal{A}_I^{\mathrm{lin}}:=\bigcup_{L\in\mathbb{N}}\{(\sigma_1\otimes\cdots\otimes\sigma_I)\circ B,\ \ \sigma_i\in\mathrm{CS}^{\mathrm{rad}}(\mathbb{R}^n),\ B\in\mathcal{M}_{I,L}(\mathbb{R}),\ \mathrm{rk}(B)=L\},$$

where $\mathrm{CS}^{\mathrm{rad}}(\mathbb{R}^n)$ stands for the algebra of classical radial symbols $\xi\mapsto\tau(|\xi|)$ with $\tau\in\mathrm{CS}(\mathbb{R}_+)$, $\mathcal{M}_{IL}(\mathbb{R})$ for the set of matrices of size $I\times L$ with coefficients in $\mathbb{R}$. The map

$$\mathcal{A}_I^{\mathrm{lin}}\times\mathcal{A}_{I'}^{\mathrm{lin}}\to\mathcal{A}_{I+I'}^{\mathrm{lin}},\ (\vec{\sigma}\circ B)\times(\vec{\sigma}'\circ B')\mapsto(\vec{\sigma}\circ B)\bullet(\vec{\sigma}'\circ B'):=(\vec{\sigma}\otimes\vec{\sigma}')\circ(B\oplus B'),$$

where $\oplus$ stands for the Whitney sum $B\oplus B':=\begin{pmatrix}B&0\\0&B'\end{pmatrix}$, induces a morphism of filtered algebras on $\mathcal{A}^{\mathrm{lin}}$.

The following is an easy consequence of Theorem 6.4.

Corollary 6.8. *Let* $\mathcal{R}\colon\tau\mapsto\tau(z)$ *be a holomorphic regularisation procedure on* $\mathrm{CS}(\mathbb{R}_+)$ *which sends a symbol* τ *of order* α *to* $\tau(z)$ *of non-constant affine order* $\alpha(z)=-qz+a$*, for some positive real number* q*. The map*

$$\Psi^{\mathcal{R}}\colon\mathcal{A}^{\mathrm{lin}}\to\mathcal{B},\quad(\vec{\sigma}\circ B)\mapsto\left(\vec{z}\mapsto\int\!\!\!\!\!\!-_{\mathbb{R}^{nL}}\vec{\sigma}(\vec{z})\circ B\right),\tag{31}$$

is a morphism of filtered algebras.

Proof. We infer from Theorem 6.4 that if $\tilde{\sigma}\circ B$ lies in $\mathcal{A}_I^{\mathrm{lin}}$ then the map $\vec{z}\mapsto⨍_{\mathbb{R}^{nL}}\vec{\sigma}(\vec{z})\circ B$ lies in $\mathcal{B}_I$. The factorisation property with respect to the product $\bullet$,

$$\Psi^{\mathcal{R}}[(\tilde{\sigma}\circ B)\bullet(\vec{\sigma}'\circ B')]=\Psi^{\mathcal{R}}(\tilde{\sigma}\circ B)\cdot\Psi^{\mathcal{R}}(\vec{\sigma}'\circ B'),$$

then follows by analytic continuation from the corresponding factorisation property on the domain of holomorphicity. □

7 Renormalised multiple discrete sums and integrals with constraints

Using generalised evaluators at zero, we now built from the two multiplicative maps $\Phi^{\mathcal{R}}\colon \mathcal{A}^{\text{cones}} \to \mathcal{B}$ (see (29)) and $\Psi^{\mathcal{R}}\colon \mathcal{A}^{\text{lin}} \to \mathcal{B}$ (see (31)) associated with a regularisation procedure $\mathcal{R}$, two characters $\mathcal{A}^{\text{cones}} \to \mathbb{C}$ and $\mathcal{A}^{\text{lin}} \to \mathbb{C}$ corresponding to their "renormalised values" at $\vec{z} = 0$. This yields renormalised sums with conical constraints and renormalised integrals with linear constraints.

7.1 Renormalised values. Combining a generalised evaluator Λ at zero on the algebra $\mathcal{B}$ defined in (8) with the maps $\Phi^{\mathcal{R}}$ and $\Psi^{\mathcal{R}}$ associated with a holomorphic regularisation $\mathcal{R}$ and using Corollaries 5.10 and 6.8 leads to the following result.

Theorem 7.1. *Given a holomorphic regularisation $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R}_+)$ and a generalised evaluator Λ at zero on the algebra $\mathcal{B}$ defined in (8), then the maps*

$$\phi^{\mathcal{R},\Lambda}\colon \mathcal{A}^{\text{cones}} \to \mathbb{C}, \quad (\vec{\sigma}; A) \mapsto \sum_{\mathcal{C}_A \cap \mathbb{Z}^k}^{\mathcal{R},\Lambda} \vec{\sigma} := \Lambda \circ \Phi^{\mathcal{R}}(\vec{\sigma}; A),$$

and

$$\psi^{\mathcal{R},\Lambda}\colon \mathcal{A}^{\text{lin}} \to \mathbb{C}, \quad (\vec{\sigma}; A) \mapsto \int_{(\mathbb{R}^n)^L}^{\mathcal{R},\Lambda} \vec{\sigma} \circ A := \Lambda \circ \Psi^{\mathcal{R}}(\vec{\sigma}; A),$$

define characters on the algebras $\mathcal{A}^{\text{cones}}$ and $\mathcal{A}^{\text{lin}}$ which extend the ordinary sums and integrals with constraints whenever these converge.

Whenever the partial sums of the orders α_i of the symbols σ_i have non-integer valued partial sums $\alpha_{\tau(1)} + \cdots + \alpha_{\tau(i)} \notin \mathbb{Z}$ for any $i \in \{1, \ldots, k\}$ and $\tau \in \Sigma_k$ (resp. $\{1, \ldots, I\}$ and $\tau \in \Sigma_I$) then

$$\sum_{\mathcal{C}_A \cap \mathbb{Z}^k}^{\mathcal{R}} \vec{\sigma} := \phi^{\mathcal{R},\Lambda}(\vec{\sigma}; A) \quad \textit{resp.} \quad \int_{(\mathbb{R}^n)^L}^{\mathcal{R}} \vec{\sigma} \circ a := \psi^{\mathcal{R},\Lambda}(\vec{\sigma}; A)$$

are independent of the evaluator Λ but might depend on the holomorphic regularisation $\mathcal{R}$.

Proof. The first part of the statement follows from the fact that $\Phi^{\mathcal{R}}$ and $\Psi^{\mathcal{R}}$ as well as the generalised evaluator Λ are multiplicative. On the domain of convergence, these maps give rise to holomorphic functions around zero on which any generalised evaluator corresponds to ordinary evaluation at zero.

The second part of the statement is a consequence of the fact that when the partial sums of the orders α_i of the symbols are non-integer valued, then the corresponding maps $\vec{z} \mapsto \Phi^{\mathcal{R}}(\vec{\sigma}(\vec{z}); A)$ and $\vec{z} \mapsto \Psi^{\mathcal{R}}(\vec{\sigma}(\vec{z}); A)$ are holomorphic around zero on which, here again, any generalised evaluator corresponds to an ordinary evaluation at zero. □

Remark 7.2. The Fubini property, i.e., invariance under a permutation of the variables in the integral and more generally invariance under a change of variables hold for renormalised integrals $-\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\Lambda} \vec{\sigma} \circ A$ just as it would hold for ordinary integrals, since it holds for the meromorphic maps $\vec{z} \mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\Lambda} \vec{\sigma}(\vec{z}) \circ A$ by analytic continuation as a consequence of the fact that these properties hold on the domain of holomorphicity. In particular, since Feynman diagrams (without external momenta) are, up to a change of variable, in one-to-one correspondence with matrices, this shows that to one diagram indeed corresponds one renormalised integral.

To sum up, we have the following multiplicativity properties:

Corollary 7.3. *Given a holomorphic regularisation* $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ *on* $\mathrm{CS}(\mathbb{R})$ *and an evaluator* Λ *on the algebra* $\mathcal{B}$ *defined in* (8)*, and for any* (σ, A), (σ', A') *in* $\mathcal{A}^{\mathrm{cones}}$*, for any* (τ, B), (τ', B') *in* $\mathcal{A}^{\mathrm{lin}}$ *we have*

$$\phi^{\mathcal{R},\Lambda}((\vec{\sigma}, A) \bullet (\vec{\sigma}', A')) = \phi^{\mathcal{R},\Lambda}(\vec{\sigma}, A) \cdot \phi^{\mathcal{R},\Lambda}(\vec{\sigma}', A'),$$
$$\psi^{\mathcal{R},\Lambda}((\vec{\tau}, B) \bullet (\vec{\tau}', B')) = \psi^{\mathcal{R},\Lambda}(\vec{\tau}, B) \cdot \psi^{\mathcal{R},\Lambda}(\vec{\tau}', B'),$$

where $\bullet$ *is the product defined in* (28)*.*

Remark 7.4. Multiplicativity corresponds to the independence of the multiple integrals and sums over disjoint sets of constraints. Although this multiplicativity property does not determine the renormalised values completely, it is a strong requirement which reflects a causality principle in physics.

7.2 Evaluators versus Birkhoff–Hopf factorisation. In the previous section, we considered algebras involving both integrands $\vec{\sigma}$ and constraints described in terms of matrices A, thus providing freedom to vary both the integrand and the constraint. According to the context, one might want to fix the type of constraint or the type of integrand:

(1) Multiple zeta functions involve a fixed type of constraint, namely Chen cones, whereas the integrands corresponding to tensor products of symbols of the type $\sigma_i(x) = x^{-s_i}\chi(x)$ (with χ a smooth cut-off function around zero) vary according to the choice of argument s_i.

(2) Multiple integrals with linear constraints arising from Feynman integrals in the absence of external momenta can (according to which physical theory one is considering) involve a fixed type of integrand given by tensor powers of some symbol (typically $\sigma(k) = \frac{1}{k^2+m^2}$ where m is the mass), the diagrams actually describing the constraints which change according to the diagram one is computing.

(3) Sums of polynomials on cones studied in [4] involve a fixed type of integrand given by a generating function but different types of conical constraints which vary with the cones.

More precisely, the morphisms $\Phi^{\mathcal{R}}$, resp. $\Psi^{\mathcal{R}}$, induce the following morphisms with values in the algebra $\mathrm{Mer}_0(\mathbb{C})$ of meromorphic functions obtained from setting $z_i = z$ and fixing the type of constraint (Chen cones) resp. the type of integrand tensor powers of some fixed symbol (e.g., $k \mapsto \sigma(k) = \frac{1}{k^2+m^2}$).

Corollary 7.5. *Let $\mathcal{R}\colon \sigma \mapsto \sigma(z)$ on $\mathrm{CS}(\mathbb{R})$ be a holomorphic regularisation.*
The map $\Phi^{\mathcal{R}}$ induces an algebra morphism

$$\begin{aligned}\widetilde{\Phi}^{\mathcal{R}}\colon \mathcal{T}(\mathrm{CS}(\mathbb{R})) &\to \mathrm{Mer}_0(\mathbb{C}),\\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \Big(z \mapsto \sum_{0<x_k\leq\cdots\leq x_1} \sigma_1(z)(x_1)\ldots\sigma_k(z)(x_k)\Big),\end{aligned}$$

on the tensor algebra $\mathcal{T}(\mathrm{CS}(\mathbb{R}))$ over $\mathrm{CS}(\mathbb{R})$. Given a radial symbol $\sigma\colon x \mapsto \tau(|x|) \in \mathrm{CS}(\mathbb{R}^d)$ and setting $\sigma(z)(x) := \tau(z)(|x|)$, the map $\Psi^{\mathcal{R}}$ induces an algebra morphism

$$\begin{aligned}\widetilde{\Psi}^{\mathcal{R}}\colon \mathcal{M}_\infty(\mathbb{R}) &\to \mathrm{Mer}_0(\mathbb{C}),\\ A &\mapsto \left(z \mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L} (\sigma(z))^{\otimes I} \circ A\right),\end{aligned}$$

on the matrix algebra $\mathcal{M}_\infty(\mathbb{R}) := \{A \in \mathrm{gl}_\infty(\mathbb{R}),\ A$ has independent column vectors$\}$, where A of rank L has I rows and L columns.

It was shown in [27] that $\mathcal{T}(\mathrm{CS}(\mathbb{R}))$ carries a graded commutative Hopf algebra structure when equipped with the stuffle product and deconcatenation coproduct. On the other hand, along the lines of [30] one can equip the set $\mathcal{C}(\mathbb{R}^\infty)$ of families $\{C_1,\ldots,C_L\}$ of column vectors[14] $C_l \in \mathbb{R}^I$, $l = 1,\ldots,L$, for some positive integer I, with a Whitney sum on matrices type product and a deconcatenation coproduct as in [30]. Implementing Birkhoff–Hopf factorisation in the two above-mentioned situations provides another pair of renormalised maps:

$$\begin{aligned}\widetilde{\phi}^{\mathcal{R}}\colon \mathcal{T}(\mathrm{CS}(\mathbb{R})) &\to \mathbb{C},\\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \sum_{0<x_k\leq\cdots\leq x_1}^{\mathcal{R},\mathrm{Birk}} \sigma_1(x_1)\ldots\sigma_k(x_k),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\phi}^{\mathcal{R}}\colon \mathcal{C}_\infty(\mathbb{R}^I) &\to \mathbb{C},\\ A = [C_1,\ldots,C_L] &\mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\mathrm{Birk}} \sigma^{\otimes I} \circ A.\end{aligned}$$

Here, as in [30], A stands for the matrix built from the tuple $(C_1,\ldots,C_L)$ of column vectors; the Fubini property then ensures that the resulting renormalised integral

[14] In [30], we considered instead tuples of column vectors seen as matrices, which unfortunately does not allow for compatibility between product and coproduct, contrarily to what we wrongly asserted in that paper.

$\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\mathrm{Birk}} \sigma^{\otimes I} \circ A$ does not depend on the choice of tuple $(C_1, \ldots, C_L)$ built from a given family $\{C_1, \ldots, C_L\}$ of column vectors. The first renormalisation method for sums on Chen cones is close in spirit to the one implemented in [18] insofar as it is based on a stuffle Hopf algebra structure. The second renormalisation is reminiscent of the renormalisation method implemented by Connes and Kreimer [8], [23] on the Hopf algebra of Feynman diagrams.

The existence of two renormalisation approaches, one by generalised evaluators and one by Birkhoff–Hopf factorisation, raises the question of how renormalisation via Birkhoff–Hopf factoristion relates with renormalisation via generalised evaluators, a question addressed in [16], [17] and studied in specific cases but which remains open in general.

7.3 An example of a renormalised sum on a cone. To a cone $\mathcal{C} = \langle v_1, \ldots, v_J \rangle \subset \mathbb{R}_+^k$ generated by vectors $\{v_j = \sum_{i=1}^k a_{ij} e_i,\ j \in \{1, \ldots, J\}\}$, we can assign a matrix $A = (a_{ij})$ with J column vectors given by the v_j's with j varying from 1 to J and set (see the notations of (29))

$$\sum_{\mathcal{C}\cap\mathbb{Z}^k}^{\mathcal{R},\Lambda} \vec{\sigma} := \phi^{\mathcal{R},\Lambda}(\vec{\sigma}, A) \quad \text{for all } \vec{\sigma} \in (\mathrm{CS}(\mathbb{R}))^{\otimes k}.$$

This expression is defined without ambiguity for a given cone. Indeed, a permutation of the column vectors does not affect the renormalised sum; this holds at the level of meromorphic extensions $\Phi^{\mathcal{R}}(\vec{\sigma}, A)$ by analytic continuation since it holds for convergent sums, so that applying the evaluator yields the same property for renormalised sums.

We can then extend these renormalised maps by linearity to any cone $\mathcal{C} \subset \mathbb{R}^k$.

By the linearity of Λ, the resulting map $\mathcal{C} \mapsto \sum_{\mathcal{C}\cap\mathbb{Z}^k}^{\mathcal{R},\Lambda} \vec{\sigma}$ is additive on sets of disjoint cones in $\mathbb{R}^k$:

$$\text{if } \mathcal{C} = \bigcup_{j=1}^J \mathcal{C}_j \text{ and } \mathcal{C}_i \cap \mathcal{C}_j = \phi \text{ if } i \neq j, \text{ then } \sum_{\mathcal{C}\cap\mathbb{Z}^k}^{\mathcal{R},\Lambda} \vec{\sigma} = \sum_{j=1}^{J} \sum_{\mathcal{C}_j\cap\mathbb{Z}^k}^{\mathcal{R},\Lambda} \vec{\sigma}.$$

Consequently, renormalised sums on Chen cones obey the stuffle relations, for the latter follow from a partition of the Chen cone into a union of disjoint subcones as explained in Theorem 4 of [27].

We refer the reader to [27] for the general case and only describe here the case of sums on two-dimensional Chen cones.

Let $\mathcal{C}^- := \mathbb{R}_+ e_1 + \mathbb{R}_+(e_1 + e_2)$, $\mathcal{C}^+ := \mathbb{R}_+ e_2 + \mathbb{R}_+(e_1 + e_2)$. The multiplicativity property of renormalised sums on disjoint sets of constraints combined with the additivity on disjoint unions of cones yields

$$\begin{aligned}
\Big(\sum_{(\mathbb{R}_+e_1)\cap\mathbb{Z}}^{\mathcal{R},\Lambda}\sigma_1\Big)\Big(\sum_{(\mathbb{R}_+e_2)\cap\mathbb{Z}}^{\mathcal{R},\Lambda}\sigma_2\Big)
&= \sum_{(C^+\cup C^-)\cap\mathbb{Z}^2}^{\mathcal{R},\Lambda}\sigma_1\otimes\sigma_2\\
&= \sum_{C^-\cap\mathbb{Z}^2}^{\mathcal{R},\Lambda}\sigma_1\otimes\sigma_2+\sum_{C^+\cap\mathbb{Z}^2}^{\mathcal{R},\Lambda}\sigma_1\otimes\sigma_2-\sum_{(C^+\cap C^-)\cap\mathbb{Z}^2}^{\mathcal{R},\Lambda}\sigma_1\otimes\sigma_2\\
&= \sum_{0<n_2\le n_1}^{\mathcal{R},\Lambda}\sigma_1(n_1\sigma_2(n_2)+\sum_{0<n_2\le n_1}^{\mathcal{R},\Lambda}\sigma_1(n_2)\sigma_2(n_1)-\sum_{0<n}^{\mathcal{R},\Lambda}(\sigma_1\,\sigma_2)(n).
\end{aligned} \tag{32}$$

Let $\sigma_s(x) = x^{-s}\chi(x)$, where χ is a smooth cut-off function with support in $\mathbb{R}_+$, which is identically one outside the unit interval, let $C_k = \{(x_1,\ldots,x_k)\in\mathbb{R}^k,\ 0 < x_k \le \cdots \le x_1\}$ and let $\sigma(z)(x) := \sigma(x)x^{-z}$; then

$$\sum_{C_k\subset\mathbb{Z}^k}\sigma_{s_1}(z_1)\otimes\cdots\otimes\sigma_{s_k}(z_k) = \sum_{0<n_k\le\cdots\le n_1} n_1^{-s_1-z_1}\ldots n_k^{-s_k-z_k}$$

is a multiple zeta function.

Remark 7.6. Equation (32) is reminiscent of the stuffle relations for multiple zeta values, but exhibiting a precise link is somewhat involved. Indeed, whereas the regularisation $\mathcal{R}$ is compatible with the tensor product, it is not compatible with the stuffle product $\sigma_1\star\sigma_2 = \sigma_1\otimes\sigma_2+\sigma_2\otimes\sigma_1+\sigma_1\sigma_2$ because of the extra product term; one needs to twist the regularisation as in [27] in order to ensure compatibility with the stuffle relations. This amounts to defining the renormalised zeta value at arguments (s_1,s_2) by $\sum_{0<n_2\le n_1}^{\mathcal{R}^*,\Lambda}\sigma_{s_1}(n_1)\sigma_{s_2}(n_2)$, where $\mathcal{R}^*$ is the twisted regularisation.

Let us evaluate $\sum_{C_-\cap\mathbb{Z}^2}\sigma_{s_1}(z_1)\otimes\sigma_{s_2}(z_2)$ at zero. By results of [16] and [27], for some integer J chosen large enough we have

$$\begin{aligned}
&\sum_{C_-\cap\mathbb{Z}^2}\sigma_{s_1}(z_1)\otimes\sigma_{s_2}(z_2)\\
&\qquad=\sum_{j=0}^{2J}B_j\,\frac{[-s_2-z_2]_{j-1}}{j!}(\zeta(s_1+s_2+z_1+z_2+j-1)-\zeta(s_1+z_1))\\
&\qquad\quad+\frac{[-s_2-z_2]_{2J+1}}{(2J+1)!}\sum_1^\infty(n^{-s_1-z_1}\int_1^n\overline{B_{2J_2+1}}(y)\,y^{-s_2-z_2-2J_2-1}dy),
\end{aligned}$$

where as before B_n is the n-th Bernoulli number, $\overline{B}_k(x) = B_k(x-[x])$ and where, for any $a\in\mathbb{R}$, we have set

$$[a]_j := a(a-1)\ldots(a-j+1)\text{ for all } j\in\mathbb{N}-\{0\},\quad [a]_0 := 1,\quad [a]_{-1} := \frac{1}{a+1}.$$

At non-positive integer arguments $s_1 = -a_1, s_2 = -a_2$ this yields

$$\begin{aligned}\sum_{C_-\cap\mathbb{Z}^2} &\sigma_{s_1}(z_1)\otimes\sigma_{s_2}(z_2)\\ &= \sum_{j=0}^{a_1+a_2+2} B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1+z_2+j-1)-\zeta(-a_1+z_1)\big)\\ &\quad+\frac{[a_2-z_2]_{a_2+2}}{(a_2+2)!}\sum_1^{\infty}\Big(n^{a_1-z_1}\int_1^n \overline{B_{a_1+a_2+3}}(y)\,y^{-2}\,dy\Big).\end{aligned}$$

The last line on the right-hand side is a holomorphic expression at zero on which all generalised evaluators at zero vanish. In order to implement the symmetrized evaluator at zero

$$\mathrm{ev}_0^{\mathrm{ren,sym}} := \tfrac{1}{2}\big(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\circ\mathrm{ev}^{\mathrm{reg}}_{z_1=0}+\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\circ\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\big),$$

we first compute

$$\begin{aligned}&\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\textstyle\sum_{C_-\cap\mathbb{Z}^2}\sigma_{-a_1}(z_1)\otimes\sigma_{-a_2}(z_2)\Big)\Big)\\ &\quad=\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\textstyle\sum_{j=0}^{a_1+a_2+2} B_j\frac{[a_2-z_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1\\ &\qquad\qquad+z_2+j-1)-\zeta(-a_1+z_1)\big)\Big)\Big)\\ &\quad=\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\textstyle\sum_{j=0}^{a_2+1} B_j\frac{[a_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1+j-1)-\zeta(-a_1+z_1)\big)\Big)\\ &\quad=\frac{1}{a_2+1}\textstyle\sum_{j=0}^{a_2+1} B_j\binom{a_2+1}{j}\big(\zeta(-a_1-a_2+j-1)-\zeta(-a_1)\big).\end{aligned}$$

This leads to

$$\begin{aligned}&\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\textstyle\sum_{C_-\cap\mathbb{Z}^2}\sigma_{-a_1}(z_1)\otimes\sigma_{-a_2}(z_2)\Big)\Big)\\ &\qquad=\tfrac{1}{a_2+1}\textstyle\sum_{j=0}^{a_2+1} B_j\binom{a_2+1}{j}\Big(-\frac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}+\frac{B_{a_1+1}}{a_1+1}\Big).\end{aligned}\tag{33}$$

We next compute

$$\begin{aligned}&\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\textstyle\sum_{C_-\cap\mathbb{Z}^2}\sigma_{-a_1}(z_1)\otimes\sigma_{-a_2}(z_2)\Big)\Big)\\ &\quad=\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\tfrac{B_0}{a_2-z_2+1}\big(\zeta(-a_1-a_2+z_1+z_2-1)-\zeta(-a_1+z_1)\big)\Big)\Big)\\ &\qquad+\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\textstyle\sum_{j=1}^{a_1+1} B_j\frac{[a_2-z_2]_{j-1}}{j!}\big(\zeta(-a_1-a_2+z_1+z_2+j-1)\\ &\qquad-\zeta(-a_1+z_1)\big)\Big)\Big)+\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\textstyle\sum_{j=a_1+2}^{a_1+a_2+2} B_j\,\frac{[a_2-z_2]_{j-1}}{j!}\\ &\qquad\cdot\big(\zeta(-a_1-a_2+z_1+z_2+j-1)-\zeta(-a_1+z_1)\big)\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\tfrac{B_0}{a_2+1}(\zeta(-a_1-a_2+z_2-1)-\zeta(-a_1))\Big)\\
&\quad + \mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\textstyle\sum_{j=1}^{a_1+1} B_j\,\tfrac{[a_2-z_2]_{j-1}}{j!}(\zeta(-a_1-a_2+z_2+j-1)-\zeta(-a_1))\Big)\\
&\quad + \textstyle\sum_{j=1}^{a_2+1} B_{j+a_1+1}\,\partial_{z_2}\Big(\tfrac{[a_2-z_2]_{j+a_1}}{(j+a_1+1)!}\Big)_{|z_2=0}\mathrm{Res}_{z_2=0}(\zeta(-a_2+z_2+j))\\
&= \tfrac{1}{a_2+1}\textstyle\sum_{j=0}^{a_2+1} B_j\binom{a_2+1}{j}(\zeta(-a_1-a_2+j-1)-\zeta(-a_1))\\
&\quad + (-1)^{a_1+1}a_1!a_2!\tfrac{B_{a_1+a_2+2}}{(a_1+a_2+2)!}
\end{aligned}
$$

since the only contribution to the residue comes from the term $j=a_1+a_2+2$. Replacing the zeta values at non-positive integers by their expressions $\zeta(-a)=-\frac{B_{a+1}}{(a+1)!}$ in terms of Bernoulli numbers yields

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{reg}}_{z_2=0}\Big(\mathrm{ev}^{\mathrm{reg}}_{z_1=0}\Big(\textstyle\sum_{C_-\cap\mathbb{Z}^2}\sigma_{-a_1}(z_1)\otimes\sigma_{-a_2}(z_2)\Big)\Big)\\
&\qquad = -\tfrac{1}{a_2+1}\textstyle\sum_{j=0}^{a_2+1} B_j\Big(\binom{a_2+1}{j}\tfrac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}\Big)+\tfrac{B_{a_1+1}}{a_1+1}\tfrac{B_{a_2+1}}{a_2+1}\\
&\qquad\quad + (-1)^{a_1+1}a_1!a_2!\,\tfrac{B_{a_1+a_2+2}}{(a_1+a_2+2)!}.
\end{aligned}
\tag{34}
$$

Note that this expression is a priori different from (33).

Combining (33) and (34) yields

$$
\begin{aligned}
&\mathrm{ev}^{\mathrm{ren,sym}}_0\Big(\textstyle\sum_{C_-\cap\mathbb{Z}^2}\sigma_{-a_1}(z_1)\otimes\sigma_{-a_2}(z_2)\Big)\\
&\qquad = -\tfrac{1}{a_2+1}\textstyle\sum_{j=0}^{a_2+1} B_j\binom{a_2+1}{j}\Big(\tfrac{B_{a_1+a_2-j+2}}{a_1+a_2-j+2}\Big)\\
&\qquad\quad + \tfrac{B_{a_1+1}}{a_1+1}\tfrac{B_{a_2+1}}{a_2+1} + (-1)^{a_1+1}a_1!a_2!\,\tfrac{B_{a_1+a_2+2}}{2(a_1+a_2+2)!}.
\end{aligned}
$$

Since Bernoulli polynomials $B_k(x)$ are of degree k in x, renormalised values at non-positive arguments $(-a_1,-a_2)$ derived in this way give rise to rational numbers.

Excipit

We have seen three instances of the issue described in the Incipit, as how a linear form $\lambda\colon \mathcal{F}\to k$ extends to a character on subalgebras $\mathcal{S}\subset\mathcal{T}(\mathcal{F})_{\mathrm{lin}}$:

- regularised evaluators at zero defined in (5),

$$\lambda = \mathrm{ev}^{\mathrm{reg}}_0 \quad \text{on } \mathcal{F} = \mathrm{Mer}_0(\mathbb{C}),$$

  which we extended to $\mathcal{B}\subset\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))_{\mathrm{lin}}$ defined in (8);
- regularised integrals introduced in Definition 3.15,

$$\lambda = \int\!\!\!\!-^{\mathcal{R}}_{\mathbb{R}^d} \quad \text{on } \mathcal{F} = \mathrm{CS}(\mathbb{R}^d),$$

which we extended to a character $\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\Lambda}$ on $\mathcal{F}^{\text{lin}} \subset \mathcal{T}(\mathcal{F})_{\text{lin}}$ in Theorem 7.1 using a holomorphic regularisation $\mathcal{R}$ and a generalised evaluator Λ;

- regularised discrete sums introduced in Definition 3.17,

$$\lambda = \sum_{\mathbb{N}}^{\mathcal{R}} \quad \text{on } \mathcal{F} = \mathrm{CS}(\mathbb{R}),$$

which we extended to a character $\int_{(\mathbb{R}^d)^L}^{\mathcal{R},\Lambda}$ on $\mathcal{A}^{\text{con}}$ in Theorem 7.1 using a holomorphic regularisation $\mathcal{R}$ and a generalised evaluator Λ.

We are still lacking the classification of such extensions.

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Spectral triples: examples and index theory

Alan L. Carey, John Phillips, and Adam Rennie

Contents

1 Introduction

The main objective of these notes is to give some intuition about spectral triples and the role they play in index theory. The notes are basically a road map, with much detail omitted. To give a complete account of all the topics covered would require at least a book, so we have opted for a sketch.

All examples of spectral triples require a lot of effort to set up, and so we have taken most of our examples from classical index theory on manifolds, where the necessary background is readily available, [50], [65]. However, we do give some examples arising from singular spaces, group algebras and so on. One more lengthy and detailed example is the graph C*-algebras, included here to motivate the semifinite extension of spectral triples.

Spectral triples are special representatives of K-homology classes for which computing the index pairing with K-theory becomes tractable. The interest of spectral triples beyond index theory arises because the construction of spectral triples invariably uses some form of geometric and/or physical input over and above their topological content as representatives of K-homology classes.

As a consequence of our choice of examples, much else has been omitted. While we will not discuss the families index theorem here, part of our motivation for developing these notes arises from the use of the Atiyah–Singer index theorem in identifying the obstructions to quantisation in gauge field theory known as anomalies. These remain one of the primary applications of index theory outside of mathematics, along with the noncommutative interpretation of the standard model of particle physics, and the work by Bellissard using noncommutative geometry techniques in the study of the quantum Hall effect (references and further details may be found in [34]). In recent applications of noncommutative geometry to number theory, index theorems also play a role, and these notes provide the basic ideas for that application as well.

These notes have always been accompanied by lectures. Sometimes the audience has been people with a geometry background, at other times the audience has been more operator theoretic. As a consequence of this, and our focus on presenting some key examples, the background assumed is rather mixed.

We assume a fair amount of differential geometry and especially pseudodifferential operator theory. We have tried to quote the main results we use, and hope our discussion can serve as a guide to those seeking to learn this subject for index purposes. On the other hand, we spend some time on Clifford algebras and the Hodge $*$-operator. We also assume some elementary theory of C*-algebras. We have tried to write the notes so that lack of C*-knowledge does not intrude too much.

There are a number of excellent textbook presentations of foundational material. Presentations of index theory on manifolds which adapt well to noncommutative geometry can be found in [10], [54], [65], [51], [48]. An introduction to noncommutative geometry and spectral triples can be found in [51], [64], [90]. More sophisticated descriptions and applications appear in [32], [34] and [39]. Noncommutative algebraic topology, that is K-theory and K-homology, are expounded very clearly in [54]. More introductory books on K-theory are [84] and [92]. For K-theory of spaces see [3]. We take the view that the noncommutative analogue of differential topology is cyclic homology and cohomology. The description in [34] remains one of the best, and further information is available in [66]. A wonderful exposition of the intertwining of spectral triples and the local index formula in noncommutative geometry is [53].

Section 2 begins by introducing the Fredholm index and Clifford algebras. Then we outline, for the Hodge–de Rham operator on a compact manifold, the sequence of arguments that leads to a well-defined Fredholm index. We then sketch the result that the index in this case is the Euler characteristic.

Throughout this sketch, we focus on those features[1] which are essential for the arguments to hold, and which can be generalised. The definition of spectral triple is then more readily seen to be a generalisation of the situation we have just studied for the Hodge–de Rham operator. We show that given a spectral triple, there is a well-defined Fredholm index arising from the data.

We finish Section 2 with a brief look at how suitable spectral triples define a metric. This highlights the geometric content of spectral triples. In addition it allows us to use metric ideas to help us construct spectral triples, and we illustrate this with some basic examples.

Section 3 returns to operators on manifolds. We define the signature operator, spin^c manifolds, and Dirac operators. The process of twisting a (Dirac-type) operator by a vector bundle to obtain a new operator is also described. We indicate how the Fredholm indices these various operators are described by the Atiyah–Singer index theorem. The section finishes with the noncommutative torus, a noncommutative example very close to the compact manifold setting we have focussed on so far.

Section 4 gives a short description of the cohomological picture of index theory, K-theory and K-homology. We show that spectral triples define K-homology classes. We then introduce the index pairing between K-theory and K-homology. This shows that a spectral triple over an algebra $\mathcal{A}$ defines a map from the K-theory of $\mathcal{A}$ to the integers.

For the examples coming from operators on manifolds, the algebra we are looking at is just $C(M)$ or $C^\infty(M)$, where M is our manifold. Amazingly, K-theory and K-homology continue to make sense for any C*-algebra, commutative or not. It is this

[1]It is here that we are most ruthless with the pseudodifferential calculus. We are interested in various properties of operators which, on a manifold, one proves to be true using the pseudodifferential calculus. For our purposes it is the properties of operators, for example Fredholmness or compactness, that are important, not how we prove that the operator has these properties. On a manifold one should, of course, use the pseudodifferential calculus to prove these various properties of differential operators.

feature that allows us to extend index theory to the noncommutative world.

The computation of the index pairing, in any practical sense, requires special properties of representatives of K-homology and K-theory classes. We conclude this section with Connes' famous formula for the Chern character of a finitely summable Fredholm module. This Chern character 'computes' the index pairing, establishes a connection with cyclic cohomology, and so gives many more tools with which index theory problems can be studied. However, for practical computations of the index pairing, the Chern character is usually not helpful.

Sections 5 and 6 aim to present formulae for the index pairing which are more practically computable. First we require a spectral triple which is regular, a notion generalising both smoothness of functions and elliptic regularity, and summable in some sense. Summability is related to dimension and integration, and we present several different flavours: finite summability, Dixmier summability, and θ-summability.

Section 5 finishes with analytic formulae for the index pairing. Like the Chern character formula, these are usually not suitable for practical calculations. Starting from these formulae and employing perturbation techniques leads to more reasonable formulae. These are the local index formula of Connes and Moscovici, and the JLO formula. Both of these formulae have an interpretation in cyclic cohomology, and Section 6 gives a brief overview of the definitions necessary to give this interpretation.

Section 7 is both more detailed and more advanced. The first five sections summarise the construction of a semifinite spectral triple for graph C^*-algebras from [70]. This provides an accessible example of semifinite noncommutative geometry as developed in [9], [17], [18].

The constructions in the first five sections also demonstrate the relationship between semifinite spectral triples and KK-theory, in analogy to the relationship between ordinary spectral triples and K-homology. Without going into Kasparov's KK-theory in too much detail, we give a general statement about this relationship, following [57].

The final section is a brief preview of current research, where traces are replaced by twisted traces. There are a number of examples where this is a natural extension of the tracial theory, but there are few general statements. This section provides some background for the paper [15] which uses graph algebras and noncommutative geometry to study Mumford curves.

Acknowledgements. We are deeply indebted to our collaborators Sergey Neshveyev, Ryszard Nest, David Pask, Aidan Sims, Fedor Sukochev and Joe Várilly. Thanks go to Roger Senior for reading an earlier version of these notes. Many thanks to Yoshiaki Maeda and Toshikazu Natsume for organising the Noncommutative Geometry and Physics workshop at Keio University, February 2009, where the third author presented lectures based on these notes. Lectures based on these notes were also presented by the third author in June 2005 at the ANU, and at the graduate school preceding the PIMS sponsored KMS and NCG workshop at the University of Victoria, July 2009. The first author also presented a series of lectures based on these notes at the Number Theory and Physics Program of the Erwin Schrödinger Institute in March–April 2009. He thanks the Erwin Schrödinger Institute for a Senior Fellowship in March–April 2009 which enabled numerous refinements and extensions of these notes to be included. Finally, we thank the referees for making numerous suggestions which have improved the exposition.

2 Preliminaries

2.1 Conventions and notations. Throughout we assume Hilbert spaces are separable and complex. The bounded linear operators on a Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$. The ideal of compact operators on $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$; it is the unique norm closed ideal in $\mathcal{B}(\mathcal{H})$. We use [87] for the theory of compact operators and Schatten ideals. The Calkin algebra is written $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. All C*-algebras in these notes are separable and complex, and unital unless otherwise stated. We will make use of some results about unbounded densely defined self-adjoint operators on Hilbert spaces and for the reader's convenience we include a brief discussion of the theory in the appendix.

A pair (M, g) means an n-dimensional compact oriented manifold M (with no boundary) equipped with a Riemannian metric g. We use X for a compact Hausdorff space and $C(X)$ is the C*-algebra of continuous functions on X. We let $\bigwedge^* M := \bigwedge^* T^*M = \bigoplus_{k=0}^n \bigwedge^k T^*M$ denote the bundle of exterior differential forms on M, and $\Gamma(\bigwedge^* M)$ the smooth sections of $\bigwedge^* M$.

In the last section we will discuss the more complicated theory of spectral triples associated to a semifinite von Neumann algebra. We do not have space to develop the requisite theory of von Neumann algebras and index theory in this context. General references to the background are Dixmier [45], Fack and Kosaki [46] and Breuer [12], [13]. A careful exposition of index theory in this framework is contained in [23].

2.2 The Fredholm index. As we explained in the introduction, we see the roots of noncommutative geometry arising in index theory. The central classical problem here is to compute an integer, called the Fredholm index, associated with certain special operators on sections of vector bundles over smooth manifolds, the elliptic pseudodifferential operators. The solution to this problem was provided by Atiyah and Singer in the 1960's, and we will discuss numerous examples and present their theorem later. In this section, we will review some definitions and fundamental results. More details may be found in the discussion of the Fredholm index in [65], which is suitably set in the context of the Atiyah–Singer index theorem.

Definition 2.1. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $F: \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator. We say that F is Fredholm if

1) $\mathrm{range}(F)$ is closed in $\mathcal{H}_2$,
2) $\ker(F)$ is finite dimensional, and

3) $\operatorname{coker}(F) := \mathcal{H}_2/\operatorname{range}(F)$ is finite dimensional.

If F is Fredholm we define

$$\operatorname{index}(F) = \dim\ker(F) - \dim\operatorname{coker}(F).$$

Example 1. The simplest example of a Fredholm operator with non-zero index is the unilateral shift operator $S\colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$. If $\{e_i : i = 1, 2 \ldots\}$ is the usual basis of $l^2(\mathbb{N})$ then the shift is defined by

$$S\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} a_i e_{i+1}, \quad a_i \in \mathbb{C}.$$

The range of S has codimension one, and is easily seen to be closed. The kernel of S is $\{0\}$, and so

$$\operatorname{index}(S) = \dim\ker(S) - \dim\operatorname{coker}(S) = 0 - 1 = -1.$$

Example 2. If $F\colon \mathcal{H} \to \mathcal{H}$ is a Fredholm operator and F is self-adjoint, then $\operatorname{index}(F) = 0$. This is because $\operatorname{coker}(F) = \ker(F^*)$.

We recall the following definition [87]:

Definition 2.2. A bounded linear operator $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if T maps any bounded sequence $\{\xi_k\}_{k\geq 0} \in \mathcal{H}_1$ to a sequence $\{T\xi_k\}_{k\geq 0} \in \mathcal{H}_2$ with a convergent subsequence.

It is a basic theorem that T is compact if and only if it is the norm limit of a sequence of finite rank operators [87]. The next result is known as Atkinson's theorem.

Proposition 2.3. *Let $F\colon \mathcal{H}_1 \to \mathcal{H}_2$. Then F is Fredholm if and only if there is an operator $S\colon \mathcal{H}_2 \to \mathcal{H}_1$ such that $FS - \mathrm{id}_{\mathcal{H}_2}$ and $SF - \mathrm{id}_{\mathcal{H}_1}$ are compact operators (on $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively). Then S is also a Fredholm operator.*

Thus the Fredholm operators $F\colon \mathcal{H} \to \mathcal{H}$ are precisely those whose image in the Calkin algebra $\mathcal{Q}(\mathcal{H})$ is invertible. Given F and S as in the Proposition, S is said to be a parametrix or approximate inverse for F.

Exercise. Show that if $F, S\colon \mathcal{H} \to \mathcal{H}$ are both Fredholm operators, then $FS\colon \mathcal{H} \to \mathcal{H}$ is also Fredholm.

We now summarise some of the important properties of the Fredholm operators.

Theorem 2.4. *Let $\mathcal{F}$ denote the set of Fredholm operators on a fixed Hilbert space $\mathcal{H}$, and let $\pi_0(\mathcal{F})$ denote the set of (norm) connected components of $\mathcal{F}$.*

(i) *The index is locally constant on $\mathcal{F}$ and induces a bijection*

$$\operatorname{index}\colon \pi_0(\mathcal{F}) \to \mathbb{Z}. \tag{2.1}$$

(ii) *The index satisfies*

$$\operatorname{index}(F^*) = -\operatorname{index}(F), \quad \operatorname{index}(FS) = \operatorname{index}(F) + \operatorname{index}(S),$$

and so the induced map (2.1) *is a group isomorphism.*

(iii) *If F is Fredholm and T is compact then $F + T$ is Fredholm and*

$$\operatorname{index}(F + T) = \operatorname{index}(F).$$

In particular, any two operators with the same index lie in the same connected component of $\mathcal{F}$ and the index is constant on these components which are open in the norm topology. Thus the index is constant under compact perturbations and also sufficiently small norm perturbations. It follows that if $\{F_t\}_{t\in[0,1]}$ is a norm continuous path of Fredholm operators, then $\operatorname{index}(F_t)$ is a constant independent of t.

We will see later that operators acting on sections of vector bundles on manifolds give rise to Fredholm operators on Hilbert spaces. The topological properties of the index will enable us to construct invariants of the underlying manifold from these operators. Surprisingly, we can frequently extend this same strategy to noncommutative spaces.

2.3 Clifford algebras. Clifford algebras play a central role in the construction and analysis of many important geometric operators on manifolds. It is worth introducing them early, as it will streamline much of what we will do. References for this material include [5], [10], [51], [65].

Let V be a finite dimensional real vector space and $(\,\cdot\,|\,\cdot\,)\colon V \times V \to \mathbb{R}$ an inner product, so

$$(v|w) = (w|v), \quad (\lambda v|w) = \lambda(v|w), \quad (v + u|w) = (v|w) + (u|w), \quad (v|v) \geq 0.$$

for $u, v, w \in V$ and $\lambda \in \mathbb{R}$. We suppose also that the inner product is nondegenerate, so that $(v|v) = 0 \implies v = 0$.

Definition 2.5. The Clifford algebra $\operatorname{Cliff}(V, (\,\cdot\,|\,\cdot\,))$ (we write $\operatorname{Cliff}(V)$ when $(\,\cdot\,|\,\cdot\,)$ is understood) is the universal unital associative algebra over $\mathbb{R}$ generated by all $v \in V$ and $\lambda \in \mathbb{R}$ subject to

$$v \cdot w + w \cdot v = -2(v|w)\mathrm{id}_{\operatorname{Cliff}(V)}.$$

Clifford algebras arise in much greater generality (see [65]), but this is enough for our purposes. Observe that if v, w are orthogonal, then in the Clifford algebra they anticommute. If we let $\bigwedge^* V = \bigoplus_{j=0}^{\dim V} \bigwedge^j V$ denote the exterior algebra of V, then we have:

Lemma 2.6. *The two algebras $\bigwedge^* V$ and* $\operatorname{Cliff}(V)$ *are linearly isomorphic* (*although not isomorphic as algebras*).

Proof. Fix an orthonormal basis $\{v_1, \ldots, v_n\}$ of the vector space V, setting the dimension to be n. We define the map $m\colon \bigwedge^* V \to \mathrm{Cliff}(V)$ by

$$m(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = v_1 \cdot v_2 \cdot \cdots \cdot v_k.$$

We leave it as an exercise to check this is an isomorphism. □

The Clifford algebra is a filtered algebra, while the exterior algebra is graded.

Exercise. Show that the exterior algebra is the associated graded algebra of the Clifford algebra.

Hence we can regard the Clifford algebra as the exterior algebra with a 'deformed' product. In [10] the map m is called a quantization map.

Exercise. Write down the inverse to the isomorphism m.

Most of the time, we work with the complexification of the Clifford algebra, $\mathbb{C}\mathrm{liff}(V) = \mathrm{Cliff}(V) \otimes \mathbb{C}$. This is because we will be using complex Hilbert spaces.

Exercise. Show that

$$\mathbb{C}\mathrm{liff}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}\mathrm{liff}(\mathbb{R}^2) = M_2(\mathbb{C}).$$

More generally we have

$$\mathbb{C}\mathrm{liff}(\mathbb{R}^k) = \begin{cases} M_{2^{(k-1)/2}}(\mathbb{C}) \oplus M_{2^{(k-1)/2}}(\mathbb{C}) & k \text{ odd}, \\ M_{2^{k/2}}(\mathbb{C}) & k \text{ even}. \end{cases}$$

The complex Clifford algebra also satisfies a universal property.

Lemma 2.7. *If A is a complex unital associative algebra and $c\colon V \to A$ is a linear map satisfying*

$$c(v)c(w) + c(w)c(v) = -2(v|w)1_A$$

for all $v, w \in V$, then there is a unique algebra homomorphism $\tilde{c}\colon \mathbb{C}\mathrm{liff}(V) \to A$ extending c.

There is a special element in the Clifford algebra which we refer to as the (complex) volume form. It is defined as follows. Suppose V is n-dimensional and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V. Define

$$\omega_{\mathbb{C}} = i^{[(n+1)/2]} e_1 \cdot e_2 \cdot \ldots e_n.$$

Then $\omega_{\mathbb{C}}^2 = 1$ and for all $v \in V$ we have

$$v \cdot \omega_{\mathbb{C}} = (-1)^{n-1} \omega_{\mathbb{C}} \cdot v.$$

If we give the Clifford algebra a complex involution or adjoint operation by setting

$$(\lambda e_1 \ldots e_k)^* = (-1)^k \bar{\lambda} e_k \ldots e_1,$$

then the Clifford algebra becomes a (finite dimensional) C*-algebra, and $\omega_{\mathbb{C}} = \omega_{\mathbb{C}}^*$.

Exercise. Check that $\omega_{\mathbb{C}}^* = \omega_{\mathbb{C}}$.

It is useful in what follows to represent the Clifford algebra by linear transformations on the exterior algebra. To do this, we need to recall the *interior product* on $\bigwedge^* V$. For $v \in V$ and $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ we define

$$v\llcorner(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} (v_i|v) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k,$$

where $\hat{}$ denotes omission. The interior product satisfies

$$v\llcorner(\varphi \wedge \psi) = (v\llcorner\varphi) \wedge \psi + (-1)^k \varphi \wedge (v\llcorner\psi), \quad \varphi \in \textstyle\bigwedge^k V,$$

and so $v\llcorner \circ v\llcorner = 0$. Also observe that for $v \in V$ and $z \in \mathbb{C}$, $(v \otimes \bar{z})\llcorner$ is the adjoint of $(v \otimes z) \wedge$ for the inner product on the complexification of $\bigwedge^* V$ (see below). Then under the isomorphism $\bigwedge^* V \cong \mathrm{Cliff}(V)$, we have

$$v \cdot \varphi = v \wedge \varphi - v\llcorner\varphi. \tag{2.2}$$

for $v \in V$ and $\varphi \in \mathrm{Cliff}(V)$. Since for all $\varphi \in \bigwedge^* V$ we have

$$v \wedge (w\llcorner\varphi) + w\llcorner(v \wedge \varphi) = (v|w)\varphi$$

one can check that the action of V on $\bigwedge^* V$ defined by the formula in eq. (2.2) satisfies

$$v \cdot w + w \cdot v = -2(v|w)\mathrm{id}_V$$

and so extends to an action of $\mathrm{Cliff}(V)$ on $\bigwedge^* V$. Similarly right multiplication by v gives

$$\varphi \cdot v = (-1)^k (v \wedge \varphi + v\llcorner\varphi), \quad \varphi \in \textstyle\bigwedge^k V.$$

Exercise. Check these relations between $\wedge$, $\llcorner$, and $\cdot$.

By associativity, the left and right actions of $\mathrm{Cliff}(V)$ commute with one another, so $\bigwedge^* V$ carries two commuting actions of the Clifford algebra. The complexification of $\bigwedge^* V$ also carries commuting representations of the complexified Clifford algebra.

Starting from the inner product on V we define a sesquilinear (bilinear if V is real) map

$$(\cdot|\cdot)^p : \textstyle\bigwedge^p V \times \bigwedge^p V \to \mathbb{R}$$

by

$$(u_1 \wedge \cdots \wedge u_p | v_1 \wedge \cdots \wedge v_p)^p := \det \begin{pmatrix} (u_1|v_1) & \cdots & (u_1|v_p) \\ \vdots & \ddots & \vdots \\ (u_p|v_1) & \cdots & (u_p|v_p) \end{pmatrix}.$$

Choose an oriented orthonormal basis $e_1, \ldots, e_n$ of V and let $\sigma = e_1 \wedge \cdots \wedge e_n$. For $\lambda \in \bigwedge^k V$ or $\lambda \in \bigwedge^k (V \otimes \mathbb{C})$, the map

$$\lambda \wedge \cdot : \textstyle\bigwedge^{n-k} V \to \bigwedge^n V,$$

is linear, and as $\bigwedge^n V$ is one-dimensional, there exists a unique $f_\lambda \in \mathrm{Hom}(\bigwedge^{n-k} V, \mathbb{R})$ such that

$$\lambda \wedge \mu = f_\lambda(\mu)\sigma \quad \text{for all } \mu \in \bigwedge^{n-k} V.$$

As $\bigwedge^{n-k} V$ is an inner product space, every such linear form is given by the inner product with a fixed element of $\bigwedge^{n-k} V$, which in this case depends on λ. Denote this element by $*\lambda$. So

$$f_\lambda(\mu) = (\mu|*\lambda)^{n-k}$$

and

$$\lambda \wedge \mu = (\mu|*\lambda)^{n-k}\sigma \quad \text{for all } \mu \in \bigwedge^{n-k} V.$$

The map

$$*\colon \bigwedge^k V \to \bigwedge^{n-k} V, \quad \lambda \mapsto *\lambda,$$

is called the *Hodge star operator*.

Lemma 2.8. *If V has a positive definite inner product and $\lambda, \mu \in \bigwedge^k V$, then*

$$*(*\lambda) = (-1)^{k(n-k)}\lambda, \quad \lambda \wedge *\mu = \mu \wedge *\lambda = (\lambda|\mu)^k\sigma.$$

This discussion of actions of the Clifford algebra on $\bigwedge^* V$ and the Hodge star operator carry over to real Clifford algebras. Now, in general, $\omega_{\mathbb{C}}$ is not an element of the real Clifford algebra (supposing V to be the complexification of a real vector space). Nevertheless, when it is in the real Clifford algebra we will see that $\omega_{\mathbb{C}}$ and the Hodge star operator are closely related.

All of these constructions extend to the exterior algebra of differential forms on a manifold M. Here we consider the vector bundle $\bigwedge^* M$ and the sections $\Gamma(\bigwedge^* M)$ with all the above operations defined pointwise. Similarly, we let $\mathbb{C}\mathrm{liff}(M)$ denote the sections of the bundle of algebras $\mathbb{C}\mathrm{liff}(T^*M, g)$, where g is a Riemannian inner product on T^*M. The complex volume form $\omega_{\mathbb{C}}$ is defined using a partition of unity and local orthonormal bases of T^*M. One can then check that $\omega_{\mathbb{C}}$ is a globally defined section of $\mathbb{C}\mathrm{liff}(M)$.

2.4 The Hodge–de Rham operator. We now have sufficient information to build our first geometric example. References for all the material in this section are [10], [65]. As usual (M, g) is a closed oriented Riemannian manifold and $L^2(\bigwedge^* M, g)$ is the Hilbert space completion of the smooth sections of the exterior bundle $\bigwedge^* T^*_{\mathbb{C}} M$ with respect to the inner product

$$\langle \omega, \rho \rangle_g = \int_M \omega \wedge *\bar{\rho}.$$

Here $*$ is the Hodge $*$-operator, described in the previous section. In this inner product, forms of different degrees are orthogonal, and the inner product is positive definite, as

$$\omega \wedge *\bar{\omega} = (\omega|\bar{\omega})\, d\mathrm{vol}.$$

The exterior derivative d extends to a closed unbounded operator on $L^2(\bigwedge^* M, g)$, [54], Lemma 10.2.1. We let d^* be the adjoint of the exterior derivative with respect to this inner product. We let $\mathcal{D} = d + d^*$, and call this the *Hodge–de Rham operator*. Since M is closed and so has no boundary, this operator is formally self-adjoint (and so symmetric) on the domain given by smooth forms, and hence by [54], Corollary 10.2.6, it extends uniquely to a self-adjoint operator on $L^2(\bigwedge^* M, g)$.

2.4.1 The symbol and ellipticity. Before analysing this example any further, we need to recall the notion of principal symbol of a differential operator $D\colon \Gamma(E) \to \Gamma(F)$ between sections of vector bundles E, F over M. Letting $\pi\colon T^*M \to M$ be the projection, the principal symbol σ_D associates to each $x \in M$ and $\xi \in T^*_x M$ a linear map $\sigma_D(x,\xi)\colon \pi^*(E_x) \to \pi^*(F_x)$ defined as follows. If D is order m and in local coordinates we have

$$D = \sum_{|\alpha|\le m} M_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \qquad \xi = \sum \xi_k dx^k \in T^*_x M,$$

then

$$\sigma_D(x,\xi) = i^m \sum_{|\alpha|=m} M_\alpha(x) \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}.$$

This local coordinate description can be pasted together to give a globally defined map

$$\sigma_D\colon T^*M \to \mathrm{Hom}(\pi^*(E), \pi^*(F)).$$

Lemma 2.9 ([54], Chapter 10). *Let D be a first order differential operator on a smooth compact manifold M. Then for $f \in C^\infty(M)$*

$$[D, f] = \sigma_D(df).$$

Proof. This is just a computation. □

Let's apply this result to the Hodge–de Rham operator. First observe that

$$\begin{aligned}(d(f\omega)|\rho) &= (df \wedge \omega|\rho) + (f d\omega|\rho)\\ &= (\omega|d\bar{f} \llcorner \rho) + (d\omega|\bar{f}\rho) \qquad (\text{since } (v\wedge)^* = \bar{v}\llcorner)\\ &= (\omega|d\bar{f} \llcorner \rho) + (\omega|d^*(\bar{f}\rho)).\end{aligned}$$

So

$$(\omega|d^*(\bar{f}\rho)) = (\omega|\bar{f} d^*\rho) - (\omega|d\bar{f}\llcorner\rho).$$

Since this is true for all forms ω, ρ and all smooth functions f, we deduce that

$$d^*(f\varphi) = fd^*\varphi - df \llcorner \varphi.$$

for all forms φ and functions f. Now for $\varphi \in \Gamma(\bigwedge^* M)$ we can compute

$$[d + d^*, f]\varphi = df \wedge \varphi + fd\varphi - df \llcorner \varphi + fd^*\varphi - fd\varphi - fd^*\varphi = df \wedge \varphi - df \llcorner \varphi.$$

Hence the principal symbol of $d + d^*$ is given by the left Clifford action on $\bigwedge^* M$. In particular, for all $f \in C^\infty(M)$, the commutator $[d + d^*, f]$ extends to a bounded operator on $L^2(\bigwedge^* M, g)$.

Much of the following relies on extending ideas based on differential operators to pseudodifferential operators. These form a wider class of operators that includes integral operators. The reason for expanding the class of operators is so that (approximate) inverses of differential operators can be treated by the same methods as differential operators. We will not cover the theory of pseudodifferential operators on manifolds in these notes, just quoting some results. The reader who wants to learn more can find many discussions, such as [50], [65], [86].

The reason we are not emphasising the pseudodifferential calculus is that the main results of this theory are proved using local or pseudolocal constructions. This means that many key results rely on carrying out estimates in local coordinates, and pasting these together over the manifold. It is precisely these local estimates which do not persist in the noncommutative examples. What does tend to generalise to the noncommutative world are the properties which have global descriptions and proofs, [53].

One result about pseudodifferential operators we need is the following.

Lemma 2.10. *Suppose that Q, $P\colon \Gamma(E) \to \Gamma(E)$ are two (pseudo)differential operators on the same vector bundle $E \to M$ of orders $q, p \geq 0$, respectively. If their principal symbols commute, then*

$$\operatorname{order}([Q, P]) \leq q + p - 1.$$

Moreover if both Q, P are differential operators, so is the commutator.

Since the operator of multiplication of differential forms by a function f has principal symbol $f\,\mathrm{id}$, which commutes with any endomorphism, we find that for a first order differential operator such as $d + d^*$, the commutator $[d + d^*, f\,\mathrm{id}]$ is order zero, namely an endomorphism.

Definition 2.11. Let E, F be complex vector bundles over the compact manifold M. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator with principal symbol $\sigma_P : T^*M \to \operatorname{Hom}(\pi^*(E), \pi^*(F))$. If for all $x \in M$ and $0 \neq \xi \in T_x^*M$ we have that $\sigma_P(x, \xi)$ is an isomorphism, we call P an *elliptic* operator.

Since $\xi \cdot \xi \cdot = -\|\xi\|^2$, where the norm is the one coming from the inner product in T_x^*M, we see that $(d + d^*)^2$ has invertible principal symbol for all $\xi \neq 0$, and so is elliptic. It then follows easily that $d + d^*$ is also elliptic.

Elliptic operators have approximate inverses, as we now describe. Let $P : \Gamma(E) \to \Gamma(E)$, where $E \to M$ is a vector bundle, be an elliptic pseudodifferential operator of order $m > 0$, say. Then there exists an elliptic operator $Q : \Gamma(E) \to \Gamma(E)$ of order $-m$ such that $PQ - \mathrm{id}$ and $QP - \mathrm{id}$ are both ‘smoothing operators’. The smoothing operators act as compact operators on $L^2(E)$, and so we have something analogous to invertibility of P modulo compacts. However, there are subtleties, as operators of positive order such as P have unbounded realisations on Hilbert space.

2.4.2 Ellipticity and Fredholm properties. We now discuss the definition of unbounded Fredholm operators such as $d + d^*$. The first problem to handle is the fact that $d + d^*$ is not a bounded operator on $L^2(\bigwedge^* M, g)$. The first difficulty unboundedness presents is that while an easy integration by parts shows that $d + d^*$ is symmetric on smooth forms, this is not the same as self-adjointness. However, every symmetric differential operator on a closed manifold is essentially self-adjoint, [54], Corollary 10.2.6, and so $d + d^*$ has a unique self-adjoint extension given by its closure. We denote this self-adjoint extension by the same symbol.

Having dealt with self-adjointness, we can use the functional calculus to introduce Sobolev spaces. These allow us to view $d + d^*$ as a bounded operator between appropriate spaces. There are other definitions of Sobolev spaces based on local constructions; however, we employ a definition employing the spectral theorem since this is what we will be forced to use in the noncommutative case.

Definition 2.12. Let M be a compact oriented n-dimensional Riemannian manifold. For $s \geq 0$, define

$$L^2_s(\textstyle\bigwedge^* M, g) = \{\xi \in L^2(\textstyle\bigwedge^* M, g) : (1 + \Delta)^{s/2}\xi \in L^2(\textstyle\bigwedge^* M, g)\},$$

where $\Delta = (d + d^*)^2$ is the Hodge Laplacian. Then $L^2_s(\bigwedge^* M, g)$ is a Hilbert space for the inner product

$$\langle \xi, \eta \rangle_s := \langle \xi, \eta \rangle + \langle (1 + \Delta)^{s/2}\xi, (1 + \Delta)^{s/2}\eta \rangle$$

and we call this the s-th Sobolev space.

Remark. This construction may be generalised to any vector bundle E by choosing a connection ∇ on E, and defining the connection Laplacian $\Delta_E := \nabla^*\nabla$, which is positive.

The point of Sobolev spaces for us is the following proposition.

Proposition 2.13. *A differential operator $D : \Gamma(\bigwedge^* M) \to \Gamma(\bigwedge^* M)$ of order $m \geq 0$ extends to a bounded operator $D : L^2_s(\bigwedge^* M) \to L^2_{s-m}(\bigwedge^* M)$ for all $s \geq m$.*

Exercise. Prove Proposition 2.13 (see [65], Proposition 2.13, for more information).

Even though the operator $d + d^*$ is Fredholm in a suitable sense, it is self-adjoint, so the index of $d + d^*$ will be zero. There is an algebraic aspect of $d + d^*$ we have neglected. If we define $\gamma : \Gamma(\bigwedge^* M) \to \Gamma(\bigwedge^* M)$ by

$$\gamma(\omega) = (-1)^k \omega, \quad \omega \in \bigwedge^k M,$$

then, since both d and d^* change the degree of a form by one, we have

$$\gamma(d + d^*) = -(d + d^*)\gamma.$$

Since $\gamma^2 = 1$, γ has ± 1 eigenvalues (on $\bigwedge^{\text{even}} M$ and $\bigwedge^{\text{odd}} M$) and we can write

$$\bigwedge^* M = \bigwedge^{\text{even}} M \oplus \bigwedge^{\text{odd}} M,\ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ d + d^* = \begin{pmatrix} 0 & (d + d^*)^- \\ (d + d^*)^+ & 0 \end{pmatrix}.$$

The differential operator $(d + d^*)^+ : \Gamma(\bigwedge^{\text{even}} M) \to \Gamma(\bigwedge^{\text{odd}} M)$ has adjoint $(d + d^*)^- : \Gamma(\bigwedge^{\text{odd}} M) \to \Gamma(\bigwedge^{\text{even}} M)$. We will see that in a suitable sense the operator $(d + d^*)^+ : L^2(\bigwedge^{\text{even}} M) \to L^2(\bigwedge^{\text{odd}} M)$ is Fredholm. Our objective then is to compute the index of $(d + d^*)^+$.

What we would like to do is define the index of $(d + d^*)^+$ to be the index of

$$(d + d^*)^+ : L^2_s(\bigwedge^{\text{even}}) \to L^2_{s-1}(\bigwedge^{\text{odd}})$$

for $s \geq 1$. However, for this to be well defined we need to know that the index is independent of s. The following is the key result linking ellipticity of pseudodifferential operators with Fredholm theory.

Theorem 2.14. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic pseudodifferential operator of order $m \geq 0$ on M. Then:*

(1) *For any open set $U \subset M$ and any $\xi \in L^2_s(E)$,*

$$P\xi|_U \in C^\infty \implies \xi|_U \in C^\infty.$$

(2) *For each $s \geq m$, P extends to a bounded Fredholm operator $P : L^2_s(E) \to L^2_{s-m}(F)$ whose index is independent of s.*

(3) *For each $s \geq m$ there is a constant C_s such that*

$$\|\xi\|_s \leq C_s(\|\xi\|_{s-m} + \|P\xi\|_{s-m}) \quad (\textit{elliptic estimate})$$

for all $\xi \in L^2_s(E)$. Hence the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|P \cdot\|_{s-m}$ on $L^2_s(E)$ are equivalent.

The key to proving this theorem is the existence of a parametrix. Once this is proved, the rest can be deduced reasonably simply; see [65] for a clear proof. The elliptic estimate also says that we can always restrict attention to operators of fixed order. Mostly we consider operators of order zero and one, for if P is of order $m > 0$ we may consider the order zero operator $P(1+\Delta)^{-m/2}$, and so on.

So now we see that the index can be defined in a sensible way. However, it is natural to ask whether it may be related to the index of a bounded linear operator on $L^2(\bigwedge^* M, g)$ without using Sobolev spaces.

Proposition 2.15. *Let $\mathcal{D}$ be a self-adjoint elliptic pseudodifferential operator on a bundle $E \to M$. Then the densely defined operator $(1+\mathcal{D}^2)^{-1/2} \colon L^2(E) \to L^2(E)$ is bounded and extends to a compact operator on $L^2(\bigwedge^* M, g)$. Hence the operator $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ is a self-adjoint Fredholm operator.*

Proof. The functional calculus for self-adjoint operators tells us that $\mathcal{D}^2$ is positive and so $1+\mathcal{D}^2$ is (boundedly) invertible. The operator $(1+\mathcal{D}^2)^{-1/2}$ maps $L^2 = L^2_0$ onto L^2_1, and is bounded in norm by at most 1, again by the functional calculus. The inclusion of L^2_1 into L^2_0 is a compact linear operator by the Rellich Lemma (this uses the compactness of M), and so $(1+\mathcal{D}^2)^{-1/2} \colon L^2 \to L^2$ is a compact operator.

The second statement follows because

$$(\mathcal{D}(1+\mathcal{D}^2)^{-1/2})^2 = \mathcal{D}^2(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}$$

and so $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ has a parametrix (itself), and so is Fredholm. □

Similarly, given a self-adjoint elliptic first order (pseudo)differential operator $\mathcal{D}$ which may be written $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$ in a fashion similar to that for the operator $d + d^*$, the index of $\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}$ is defined and equals that of all the closed extensions of $\mathcal{D}$ on Sobolev spaces by Proposition 2.15. Before discussing the index of $(d+d^*)^+$, we quote one further result about elliptic differential operators.

Theorem 2.16 ([65], Theorem 5.5). *Let $P \colon \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint differential operator over a compact Riemannian manifold. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(E) = \ker P \oplus \operatorname{im} P.$$

2.4.3 The index of the Hodge–de Rham operator. To determine the index of $(d + d^*)^+$, we are going to need a little more machinery. Let $\Delta = (d+d^*)^2$ be the Laplacian on forms, and observe that $\Delta = dd^* + d^*d$. Then $\operatorname{im}(\Delta) = \operatorname{im}(d) + \operatorname{im}(d^*)$ and so by Theorem 2.16 we have:

Proposition 2.17 (The Hodge decomposition theorem). *Let M be a compact oriented Riemannian manifold, and let $\boldsymbol{H}^p$ denote the kernel of $\Delta = (d+d^*)^2$ on p-forms. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(\textstyle\bigwedge^p M) = \boldsymbol{H}^p \oplus \operatorname{im}(d) \oplus \operatorname{im}(d^*), \quad p = 0, \dots, n. \tag{2.3}$$

In particular, there is an isomorphism

$$\boldsymbol{H}^p \cong H^p_{dR}(M;\mathbb{R}), \quad p = 0, \dots, n,$$

where $H^p_{dR}(M;\mathbb{R})$ *denotes the* p*-th de Rham cohomology group.*

Proof (Sketch). The first statement follows directly from Theorem 2.16 and the fact that $d^2 = d^{*2} = 0$. For the second, we observe that eq. (2.3) says

$$\boldsymbol{H}^p \oplus \mathrm{im}(d) = \mathrm{coker}(d^*) = \ker(d).$$

Hence

$$H^p_{dR}(M;\mathbb{R}) = \frac{\ker(d)}{\mathrm{im}(d)} = \boldsymbol{H}^p.$$

□

From this result it is now not too difficult to prove the following:

Theorem 2.18. *The index of* $(d + d^*)^+$ *is*

$$\begin{aligned}\mathrm{index}(d + d^*)^+ &= \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M;\mathbb{R}) \\ &= \chi(M) = \textit{the Euler characteristic of } M \\ &= \textit{a homotopy invariant of } M \\ &= \textit{independent of the metric } g.\end{aligned}$$

These index calculations depend on the analysis of pseudodifferential operators, which we have omitted. In particular, it is the pseudodifferential machinery which allows us to see that for P elliptic, $Pu \in C^\infty \implies u \in C^\infty$. This has the corollary that all elements of the kernel and cokernel of $(d + d^*)^+$ are smooth. In some sense it is this fact that tells us that the index is independent of s. For an introduction to this pseudodifferential theory, see [65] or [48].

2.5 The definition of a spectral triple.

The example of the previous section motivates the following notions.

Definition 2.19. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$, that is,

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \quad \pi \text{ is a } *\text{-homomorphism}$$

along with a densely defined self-adjoint (typically unbounded) operator

$$\mathcal{D} : \ \mathrm{dom}\, \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$$

satisfying the following conditions.

(1) For all $a \in \mathcal{A}$ we have $\pi(a)\,\mathrm{dom}\,\mathcal{D} \subset \mathrm{dom}\,\mathcal{D}$, and the densely defined operator $[\mathcal{D}, \pi(a)] := \mathcal{D}\pi(a) - \pi(a)\mathcal{D}$ is bounded (and so extends to a bounded operator on all of $\mathcal{H}$ by continuity).

(2) For all $a \in \mathcal{A}$ the operator $\pi(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator.

If in addition there is an operator $\gamma \in \mathcal{B}(\mathcal{H})$ with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$, and for all $a \in \mathcal{A}$ we have $\gamma\pi(a) = \pi(a)\gamma$, then we call the spectral triple *even* or *graded*. Otherwise it is *odd* or *ungraded*.

Remarks. This appears to be an unwieldy definition. There are numerous ingredients and using unbounded operators makes things technically more difficult. However, as we have seen, this is actually the structure one encounters naturally when doing index theory, and has computational advantages, as we shall see. We will nearly always dispense with the representation π, treating $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$.

In the sequel we will make a general simplifying assumption unless explicitly mentioned otherwise. Namely, in all spectral triples the algebra $\mathcal{A}$ is unital; i.e., there is $1 \in \mathcal{A}$ such that $1a = a1 = a$, $1^* = 1$. If $1 \in \mathcal{A}$ acts as the identity of the Hilbert space, then this implies in particular that $(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator. We will assume in the following that even if $1 \in \mathcal{A}$ does not act as the identity, we always have $(1 + \mathcal{D}^2)^{-1/2}$ compact.

Example 3. The 'Hodge–de Rham' triple $(C^\infty(M), L^2(\bigwedge^* M, g), d + d^*)$ of an oriented compact manifold M is an example of a spectral triple (we have in fact already verified this). It is always even, being graded by the degree of forms modulo 2.

Example 4. Let $\mathcal{H} = L^2(S^1)$ and $\mathcal{A} = C^\infty(S^1)$ act by multiplication operators on $\mathcal{H}$. The one-dimensional Dirac operator (see next section) is

$$\mathcal{D} = \frac{1}{i}\frac{d}{d\theta},$$

where we are using local coordinates to define $\mathcal{D}$. This is an odd spectral triple, as a little Fourier theory will reveal. This is a useful example to understand in detail.

2.5.1 The Fredholm index in a spectral triple. We want to study index theory in a spectral triple by analogy with the example of the Hodge–de Rham operator. First we need to show that the unbounded operator $\mathcal{D}$ appearing in the spectral triple has a well-defined index and for this we need the notion of an unbounded Fredholm operator. To this end we form the Hilbert space $\mathcal{H}_1 = \{\xi \in \mathcal{H} : \mathcal{D}\xi \in \mathcal{H}\}$ with the inner product

$$\langle \xi, \eta \rangle_1 := \langle \xi, \eta \rangle + \langle \mathcal{D}\xi, \mathcal{D}\eta \rangle$$

Then $\mathcal{D}$ is a bounded operator from $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}$ (**Exercise**). We will say that $\mathcal{D}$ is an unbounded Fredholm operator if $\mathcal{D}\colon \mathcal{H}_1 \to \mathcal{H}$ is a Fredholm operator in the sense of Definition 2.1.

Lemma 2.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Then $\mathcal{D}$ is unbounded Fredholm.*

Proof. To see this, we produce an inverse up to compacts. Such an approximate inverse (parametrix) is given by

$$\mathcal{D}(1+\mathcal{D}^2)^{-1} : \mathcal{H} \to \mathcal{H}_1$$

since

$$\mathcal{D} \cdot \mathcal{D}(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}. \qquad \square$$

Exercise. Fill in the details of this proof.

Since $\mathcal{D}$ is self-adjoint, it has zero index, but when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even, or graded, we define

$$\mathcal{D}^+ = \tfrac{1-\gamma}{2}\mathcal{D}\tfrac{1+\gamma}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^-.$$

For an even spectral triple, this is the operator of interest, and it too is Fredholm since $\mathcal{D}^+\mathcal{D}^-(1+\mathcal{D}^2)^{-1}$ is 'almost' the identity on $\mathcal{H}_-$. Since $\mathcal{D}$ is unbounded, we need to be careful about what we mean by the index. This is analogous to using Sobolev spaces for elliptic operators on manifolds.

Definition 2.21. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. For $s \geq 0$ define $\mathcal{H}_s = \{\xi \in \mathcal{H} : (1+\mathcal{D}^2)^{s/2}\xi \in \mathcal{H}\}$. With the inner product

$$\langle \xi, \eta \rangle_s := \langle \xi, \eta \rangle + \langle (1+\mathcal{D}^2)^{s/2}\xi, (1+\mathcal{D}^2)^{s/2}\eta \rangle,$$

$\mathcal{H}_s$ is a Hilbert space. Finally let

$$\mathcal{H}_\infty := \bigcap_{s\geq 0} \mathcal{H}_s = \bigcap_{s \geq 0} \operatorname{dom}(1+\mathcal{D}^2)^{s/2}.$$

Corollary 2.22. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an even spectral triple with grading γ. Write $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$, and let $\mathcal{D}_s^+$ be the restriction $\mathcal{D}^+ : \mathcal{H}_s \to \mathcal{H}_{s-1}$, where $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{H}_1 = \operatorname{dom}(\mathcal{D})$. For all $s \geq 1$ we have*

$$\operatorname{index}(\mathcal{D}_s^+) = \operatorname{index}(\mathcal{D}^+) = \operatorname{index}(\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}),$$

where the middle index is of $\mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^-$ and the last is the index of the bounded operator $\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2} : \mathcal{H}^+ \to \mathcal{H}^-$.

Proof. Suppose that $\mathcal{D}\xi = \lambda\xi$, so that ξ is an eigenvector. Then, since $\xi \in \operatorname{dom} \mathcal{D}$, we see that $\xi \in \mathcal{H}_\infty$. So all the eigenvectors of $\mathcal{D}$ lie in $\mathcal{H}_\infty$, and in particular, if $\mathcal{D}\xi = 0$, $\xi \in \mathcal{H}_\infty$. Consequently, if $\mathcal{D}^+\xi = 0$, $\xi \in \mathcal{H}_\infty^+$, and similarly for $\mathcal{D}^-$. Hence the kernel and cokernel of $\mathcal{D}^+$ consist of elements of $\mathcal{H}_\infty$, and the index is independent of which 'Sobolev space' we use. The equality with $\operatorname{index}(\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2})$ now follows from the invertibility of $(1+\mathcal{D}^2)^{-1/2}$. $\square$

Example 5. In finite dimensions, i.e., $\dim \mathcal{H} < \infty$, we can take $\mathcal{A}$ to be finite dimensional, and so we are dealing with sums of matrix algebras. There is then really no constraint in the definition of spectral triple. If we have an even triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ which is finite dimensional in this sense, then

$$\operatorname{index}(\mathcal{D}_+ \colon \mathcal{H}_+ \to \mathcal{H}_-) = \dim \mathcal{H}_+ - \dim \mathcal{H}_-$$

by the rank nullity theorem.

2.5.2 Connes' metric for spectral triples. Here we mention one important geometric feature of spectral triples, the metric on the state space of the algebra. This is of some importance for the construction of spectral triples for particular algebras. The heuristic idea that the operator '$\mathcal{D}$' of a spectral triple is some sort of differentiation allows us to use metric ideas to construct '$\mathcal{D}$', so it is compatible with the notion of difference quotients coming from a given metric. We give examples below.

A state on a unital C*-algebra is a linear functional $\phi \colon A \to \mathbb{C}$ with $\phi(1) = 1 = \|\phi\|$ and $\phi(a^*a) \geq 0$ for all $a \in A$. This is a convex space, and the extreme points (those states that are not convex combinations of other states) are called pure states. We denote the state space by $\mathcal{S}(A)$ and the pure states by $\mathcal{P}(A)$. The pure states of a commutative C*-algebra, $C(X)$, correspond to point evaluations. So, for $x \in X$, defining $\phi_x(f) = f(x)$ for all $f \in C(X)$ gives a pure state, and they are all of this form. Indeed, the weak-* topology on $\mathcal{P}(C(X))$ is the original topology on X, and $\mathcal{P}(C(X)) \simeq X$ (Gelfand Theory).

We now show that spectral triples are 'noncommutative metric spaces'. We begin with the definition of the metric.

Lemma 2.23. *Suppose that* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a spectral triple such that*

$$\{a \in \mathcal{A} \setminus \{\mathbb{C} \cdot 1\} : \|[\mathcal{D}, a]\| \leq 1\} \tag{2.4}$$

is a norm bounded set in the Banach space $\bar{\mathcal{A}}/\mathbb{C} \cdot 1$. *Then*

$$d_{\mathcal{D}}(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[\mathcal{D}, a]\| \leq 1\}$$

defines a metric on $\mathcal{P}(\bar{\mathcal{A}})$*, the pure state space of* $\bar{\mathcal{A}}$.

Proof. The triangle inequality is a direct consequence of the definition. To see that $d(\phi, \psi) = 0$ implies $\phi = \psi$, suppose that $\phi \neq \psi$. Then there is some $a \in \bar{\mathcal{A}}$ with $\phi(a) \neq \psi(a)$, and we can use the density of $\mathcal{A}$ in $\bar{\mathcal{A}}$ to find an element of $b \in \mathcal{A}$ such that $\phi(b) \neq \psi(b)$, and so $d(\phi, \psi) \neq 0$. The norm boundedness of the set in (2.4) gives the finiteness of the distance between any two pure states. □

In the future, when we mention the metric associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we implicitly assume that the condition (2.4) is met. In particular, it means that no element of $\mathcal{A}$ except scalars commutes with $\mathcal{D}$.

The metric is actually defined on the whole state space $\mathcal{S}(A)$, but the metric on $\mathcal{S}(A)$ need not be determined by the restriction of the metric to the pure states, even for commutative examples, [82]. Much more information about 'compact quantum metric spaces' is contained in [81], [83], [82] and references therein. In particular, Rieffel proves that if the set in eq. (2.4) is in fact pre-compact in $\bar{\mathcal{A}}/\mathbb{C}\cdot 1$, then the metric induces the weak-* topology on the state space.

Note that when $\mathcal{A}$ is commutative, so that $\mathcal{A}$ is an algebra of (at least continuous for the weak-* topology) functions on $X = \mathcal{P}(\bar{\mathcal{A}})$, the metric topology on $\mathcal{P}(\bar{\mathcal{A}})$ is automatically finer than the weak-* topology. In the case of a smooth Riemannian spin manifold, whose algebra of smooth functions is finitely generated by the (local) coordinate functions, not only do the topologies on the pure state space agree, so do the metrics, [34], [33].

Lemma 2.24. *If $(C^\infty(M), L^2(E), \mathcal{D})$ is the spectral triple of any 'Dirac type operator' of the Clifford module E on a compact Riemannian spin manifold (M, g), then*

$$d_{\mathcal{D}}(\phi,\psi) = d_\gamma(\phi,\psi) \quad \text{for all } \phi,\ \psi \in \mathcal{P}(C(M)),$$

where d_γ is the geodesic distance on X.

For a discussion of general Dirac type operators, see [10], [65]. It is enough for us that $d + d^*$ and the Dirac operator of a spin or spin^c structure (discussed in the next section) are of Dirac type.

More generally, whenever $\mathcal{A}$ is commutative, we can take $\mathcal{A} \subseteq \mathrm{Lip}_d(\mathcal{P}(\mathcal{A}))$, the Lipschitz functions with respect to the metric topology since

$$|a(\phi) - a(\psi)| := |\phi(a) - \psi(a)| \le \|[\mathcal{D}, a]\| d(\phi,\psi)$$

for all $a \in \mathcal{A}$, $\phi, \psi \in \mathcal{P}(\mathcal{A})$.

Example 6. Here is a simple way to use metric ideas to build a spectral triple. Let $\mathcal{A} = \mathbb{C}\oplus\mathbb{C}$ be the continuous functions on two points. Let $\mathcal{A}$ act on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ by multiplication. Let $0 \neq m \in \mathbb{R}$ and set $\mathcal{D} = \begin{pmatrix} 0 & m \\ m & 0\end{pmatrix}$ with the grading $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. The index here is zero, but the distance between the two points is $\frac{1}{m}$. Check this as an

Exercise. If we want a nonzero index as well, let $(a, b) \in \mathcal{A}$ act on $\mathbb{C}^3$ by

$$\pi(a,b)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_2\end{pmatrix} = \begin{pmatrix} a\xi_1\\ b\xi_2\\ b\xi_3\end{pmatrix}$$

and define $\mathcal{D} = \begin{pmatrix} 0_2 & \bar{m} \\ (\bar{m})^T & 0\end{pmatrix}$ where now $\bar{m} = \begin{pmatrix} m \\ 0\end{pmatrix}$ and $\gamma = 1 \oplus -1_2$. As a further **Exercise** calculate the distance and the index for this example.

These simple cases lead quickly to more complicated examples of similar type with different indices and different numbers of points and so on. However, as the number of points goes up, the expression for the distance (given a generic operator $\mathcal{D}$) becomes more and more complicated. It has been shown that for a particular class of examples of this form, polynomials of degree 5 and more arise, so the distance is generically not computable using arithmetic and the extraction of roots, [55].

Another level of complexity is added when we consider matrix algebras instead of copies of $\mathbb{C}$. This is because we can have much more complicated commutation relations. We refer to [55] for a fuller discussion of these examples, but recommend that the interested reader first gain some experience by computing special cases. Note that these are *not* just toy models. Taking the product of the Dirac spectral triple of a manifold with certain spectral triples for sums of matrix algebras yields spectral triples with close relationships with particle physics. The reconstruction of the (classical) Lagrangian of the standard model of particle physics from such a procedure is a fundamental example. See [91] for an introduction and a guide to some of the extensive literature on this subject. For more information on these finite spectral triples, see [55], [61], [69].

There are many other examples of spectral triples built with the intention of recovering or studying metric data (also dimension type data). Some interesting examples are contained in the recent papers of Erik Christensen and Cristina Ivan, as well as their coauthors. See [31], [30], [29], [1]. We present one more example here which we will look at again when we come to dimensions.

Example 7. This example illustrates how to use metric ideas to construct a spectral triple. Many thanks to Nigel Higson for relating it to us. Consider the Cantor 'middle thirds' set K. So start with the unit interval, and remove the (open) middle third. Then remove the open middle third of the two remaining subintervals, etc. Observe that end points of removed intervals come in pairs, e_-, e_+, where e_- is the left endpoint of a gap and e_+ is the right endpoint of a gap. Every end point except 0, 1 is one (and only one) of these two types, and we take 0, 1 as a pair.

Let $\mathcal{H} = l^2(\text{end points})$ and $\mathcal{A}$ be the locally constant functions on K. Recall that a function $f\colon K \to \mathbb{C}$ is locally constant if for all $x \in K$ there is a neighbourhood U of x such that f is constant on U. Define $\mathcal{D}\colon \mathcal{A} \cap \mathcal{H} \to \mathcal{H}$ by

$$(\mathcal{D} f)(e_+) = \frac{-if(e_-)}{e_+ - e_-}, \quad (\mathcal{D} f)(e_-) = \frac{if(e_+)}{e_+ - e_-}.$$

The closure of this densely defined operator is self-adjoint (**Exercise**). Also

$$[\mathcal{D}, f]g(e_+) = i\,\frac{f(e_+) - f(e_-)}{e_+ - e_-}g(e_-), \quad [\mathcal{D}, f]g(e_-) = i\,\frac{f(e_+) - f(e_-)}{e_+ - e_-}g(e_+).$$

For f locally constant the commutator $[\mathcal{D}, f]$ defines a bounded operator.

Let δ_{e_+} be the function which is one on e_+ and zero elsewhere, and similarly for δ_{e_-}. Now these are not locally constant functions, but are in the domain of (the closure of) $\mathcal{D}$. We observe that $\text{span}\{\delta_{e_+}, \delta_{e_-}\}$ is invariant under $\mathcal{D}$ since

$$\mathcal{D}\delta_{e_+} = \frac{i\delta_{e_-}}{e_+ - e_-}, \quad \mathcal{D}\delta_{e_-} = -\frac{i\delta_{e_+}}{e_+ - e_-}.$$

Indeed, in the basis given by δ_{e_+}, δ_{e_-}, $\mathcal{D} = \frac{i}{e_+ - e_-}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence $\mathcal{D}$ has eigenvalues $\pm 1/(e_+ - e_-) = \pm 3^n$ if the points appear in the n-th stage of the construction, and their multiplicity is 2^{n-1}. Thus $(1 + \mathcal{D}^2)^{-1}$ is certainly compact and invertible.

It is now an **Exercise** to show that Connes' metric is precisely the usual metric on the Cantor set. We will return to this example when we discuss summability.

3 More spectral triples from manifolds

Our aim in this section is to produce more examples of spectral triples from the theory of elliptic differential operators on manifolds. To achieve this we will need a little more differential geometry, as well as the Clifford algebra formalism we have introduced previously.

3.1 The signature operator. In dimensions $n = 4k$ there is another grading on the space $\Gamma(\bigwedge^* T^*M)$ that allows us to define a new spectral triple. In these dimensions, the complex volume form $\omega_{\mathbb{C}}$ is given by $(-1)^k\omega = (-1)^k e_1 \dots e_{4k}$ in terms of a local orthonormal basis. Consequently, the Clifford action by $\omega_{\mathbb{C}}$ maps the space of real sections of $\bigwedge^* M$ into itself. Moreover, for $\varphi \in \bigwedge^p M$ we have

$$\omega_{\mathbb{C}} \cdot \varphi = (-1)^{k+p(p-1)/2} * \varphi,$$

where $*$ is the Hodge star operator. Since (in even dimensions) $d^* = - * d *$, we see that $d + d^*$ anticommutes with the action of $\omega_{\mathbb{C}}$, and we get a new grading. We already know that $d + d^*$ has compact resolvent, so when $\dim M = n = 4k$,

$$(C^\infty(M), L^2(\textstyle\bigwedge^* M, g), d + d^*, \omega_{\mathbb{C}})$$

is an even spectral triple. Now we ask: what is the index? The answer takes several steps.

Claim: The identification up to sign of $\omega_{\mathbb{C}}\cdot$ and $*$ gives us isomorphisms

$$\omega_{\mathbb{C}} \colon \boldsymbol{H}^p \to \boldsymbol{H}^{4k-p}$$

for each $p = 0, 1, \dots, 2k$.

Proof. We know from the Hodge decomposition theorem that $\varphi \in \boldsymbol{H}^p$ if and only if $d\varphi = d^*\varphi = 0$. So let $\varphi \in \boldsymbol{H}^p$. Then $d\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}} d^*\varphi = 0$ and similarly, $d^*\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}} d\varphi = 0$. Since $\omega_{\mathbb{C}}^2 = 1$, we are done. □

So for $p = 0, \dots, 2k-1$ the space $\boldsymbol{H}(p) = \boldsymbol{H}^p \oplus \boldsymbol{H}^{4k-p}$ has a decomposition

$$\boldsymbol{H}(p) = \boldsymbol{H}^+(p) \oplus \boldsymbol{H}^-(p) = \tfrac{(1+\omega_{\mathbb{C}})}{2}\boldsymbol{H}(p) \oplus \tfrac{(1-\omega_{\mathbb{C}})}{2}\boldsymbol{H}(p).$$

Claim: The subspaces $\frac{1}{2}(1 \pm \omega_{\mathbb{C}})\boldsymbol{H}(p)$ have the same dimension.

Proof. Let $\{\varphi_1, \dots, \varphi_m\}$ be a basis of $\boldsymbol{H}^p$. Then the subspace $\boldsymbol{H}^+(p)$ has basis $\{\varphi_1 + \omega_{\mathbb{C}}\varphi_1, \dots, \varphi_m + \omega_{\mathbb{C}}\varphi_m\}$. Likewise $\boldsymbol{H}^-(p)$ has basis $\{\varphi_1 - \omega_{\mathbb{C}}\varphi_1, \dots, \varphi_m - \omega_{\mathbb{C}}\varphi_m\}$. □

This allows us to compute

$$\ker \mathcal{D}^\pm = \boldsymbol{H}^\pm := \boldsymbol{H}^\pm(0) \oplus \boldsymbol{H}^\pm(1) \oplus \cdots \oplus \boldsymbol{H}^\pm(2k-1) \oplus (\boldsymbol{H}^{2k})^\pm,$$

where $(\boldsymbol{H}^{2k})^\pm = (1 \pm \omega_{\mathbb{C}})\boldsymbol{H}^{2k}$. Since the index is $\dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^-$, we have

$$\operatorname{index}(d + d^*) = \dim(\boldsymbol{H}^{2k})^+ - \dim(\boldsymbol{H}^{2k})^-.$$

Observe that, on $\bigwedge^{2k} M$, $\omega_{\mathbb{C}}$ and the Hodge star operator coincide. We define a new inner product on $\boldsymbol{H}^{2k}$ by

$$\langle \phi, \psi \rangle^{\text{new}} := \int_M \phi \wedge \psi.$$

Then, using $* = \omega_{\mathbb{C}}$ and $\int_M \phi \wedge *\phi = \|\phi\|^2$, we see that the signature of this inner product (the number of positive eigenvalues minus the number of negative eigenvalues) is precisely the difference in dimensions of the ± 1 eigenspaces of the $*$ operator. So, letting signature(M) denote the signature of this inner product we have our answer:

$$\operatorname{index}((d + d^*)^+, \omega_{\mathbb{C}}) = \operatorname{signature}(M).$$

Remarkably, this is a homotopy invariant of the manifold.

3.2 Connections and twistings. To go further than the preceding section it is very useful to make much of our discussion more streamlined by exploiting Clifford algebras. Recall that if W is a left module over a $*$-algebra $\mathcal{A}$, a Hermitian form is a map $(\cdot\,|\,\cdot)\colon W \times W \to \mathcal{A}$ such that

$$(v + w|u) = (v|u) + (w|u), \quad (av|u) = a(v|u), \quad (v|u) = (u|v)^*$$

for all $u, v, w \in W$ and $a \in \mathcal{A}$. If $\mathcal{A}$ is a pre-C*-algebra, we can also ask for $(v|v) \geq 0$ in the sense of the C*-closure of $\mathcal{A}$, and that $(v|v) = 0 \implies v = 0$. We assume that all our Hermitian forms satisfy these properties.

For a vector bundle $E \to M$, we know that $\Gamma(E)$ is a module over $C^\infty(M)$ via $(f\sigma)(x) = f(x)\sigma(x)$ for all $f \in C^\infty(M)$, $\sigma \in \Gamma(E)$ and $x \in M$. A Hermitian form is then a collection of positive definite inner products $(\cdot\,|\,\cdot)_x$ on E_x such that for all smooth sections $\sigma, \rho \in \Gamma(E)$, the function $x \to (\sigma(x)|\rho(x))_x$ is smooth. All complex vector bundles have such a smooth inner product.

Now let $E \to M$ be a smooth vector bundle, and let $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection. So ∇ is $\mathbb{C}$-linear and

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. We can extend ∇ to a map $\nabla\colon \Gamma(\bigwedge^* M \otimes E) \to \Gamma(\bigwedge^* M \otimes E)$ by defining

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \otimes \nabla\sigma$$

for $\omega \in \Gamma(\bigwedge^k M)$ and $\sigma \in \Gamma(E)$.

Claim: The operator ∇^2 is linear over $C^\infty(M)$.

Proof. Let $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. Then

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = d^2 f \otimes \sigma - df \otimes \nabla\sigma + df \otimes \nabla\sigma + f\nabla^2\sigma = f\nabla^2\sigma.$$

□

Thus ∇^2 is a two-form with values in the endomorphisms of E (locally a matrix of two-forms). It is called the curvature of E.

A connection ∇ on a vector bundle $E \to M$ with Hermitian form $(\cdot|\cdot)$ is said to be compatible with $(\cdot|\cdot)$ if

$$d(\sigma|\rho) = (\nabla\sigma|\rho) + (\sigma|\nabla\rho)$$

for all $\sigma, \rho \in \Gamma(E)$, where to interpret the right-hand side we write in local coordinates $\nabla\sigma = \sum_i dx^i \otimes \sigma_i$ and $\nabla\rho = \sum_j dx^j \otimes \rho_j$ and then

$$(\nabla\sigma|\rho) + (\sigma|\nabla\rho) := \sum_i dx^i (\sigma_i|\rho) - \textstyle\sum_j dx^j (\sigma|\rho_j).$$

Example 8. The Levi-Civita connection on TM or T^*M is compatible with the Riemannian metric. The curvature of the Levi-Civita connection is, by definition, the curvature of the manifold.

Lemma 3.1 (see [65]). *Let M be a compact oriented manifold, and let c denote the usual left action of* $\mathbb{C}\mathrm{liff}(M)$ *on* $\bigwedge^* M$. *Let ∇ denote the Levi-Civita connection on T^*M. Then*

$$d + d^* = c \circ \nabla.$$

Thus the Hodge–de Rham and signature operators are both given by composing the Levi-Civita connection with the Clifford action. This is a very general recipe, and allows us to construct versions of these operators that are ‘twisted’ by a vector bundle.

If $E, F \to M$ are both vector bundles, with connections ∇^E, ∇^F respectively, we can define a connection $\nabla^{E,F}$ on the tensor product $E \otimes F$ by defining

$$\nabla^{E,F}(\sigma \otimes \rho) = (\nabla^E \sigma) \otimes \rho + \sigma \otimes (\nabla^F \rho)$$

for all $\sigma \in \Gamma(E)$ and $\rho \in \Gamma(F)$. If E is a $\mathbb{C}\mathrm{liff}(M)$ module, then so is $E \otimes F$ by letting $\mathbb{C}\mathrm{liff}(M)$ act only on E. Provided that for all $\sigma \in \Gamma(E)$ and $\varphi \in \Gamma(T^*M)$ we have

$$\nabla^E(c(\varphi)\sigma) = c(\nabla^{T^*M}\varphi)\sigma + c(\varphi)\nabla^E\sigma,$$

we can form the operator

$$c \circ \nabla^{E,F} : \Gamma(E \otimes F) \to \Gamma(E \otimes F),$$

where c denotes the Clifford action. Choose a Hermitian structure on E so that for any one-form φ

$$(c(\varphi)\rho|\sigma) = -(\rho|c(\bar{\varphi})\sigma), \quad \rho, \sigma \in E,$$

where c denotes the Clifford action. Such an inner product always exists. The choice of inner product and connection ensures that $c \circ \nabla^{E,F}$ is (formally) self-adjoint.

Applying this recipe to $d + d^*$ allows us to 'twist' $d + d^*$ by any vector bundle

$$(d + d^*) \otimes_\nabla \mathrm{id}_E := c \circ \nabla^{T^*M,E}.$$

By choosing a connection compatible with a product Hermitian structure, and using the integral to define a scalar inner product, we can construct a new spectral triple

$$(C^\infty(M), L^2(\textstyle\bigwedge^* M \otimes E, g \otimes (\cdot \mid \cdot)), (d + d^*) \otimes_\nabla \mathrm{id}_E).$$

Remark. We have made use of the commutativity of the algebra at a few points in the above discussion. For example, we have identified the right and left actions of functions on sections by multiplication. This allows us to use $\Gamma(E \otimes F) = \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

Many of these arguments are unavailable in the noncommutative case. If we have a spectral triple $(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}^{\mathrm{op}}$ is the opposite algebra, then we can twist by finite projective modules $p\mathcal{A}^N$ or equivalently $p\mathcal{A}^{\mathrm{op},N}$, and be left with a spectral triple for one copy of $\mathcal{A}$. This point of view underlies Poincaré duality in K-theory.

3.3 The spin^c condition and the Dirac operator. Building differential operators from connections and Clifford actions yields operators which depend on the Riemannian metric and the Clifford module. The index of such an operator is an invariant of the manifold and the Clifford module, [10], Theorem 3.51. Since for both the Hodge–de Rham and signature operator the Clifford module depends only on the manifold, we see that the Euler characteristic and the signature are invariants of the manifold. Thus it is natural to ask what other operators can one build in this way. For spin and spin^c manifolds, there is an interesting answer. The spin^c condition was originally formulated in differential geometry language, involving double covers of principal SO bundles on a manifold. This brings in the spin groups and their representations. This will take us too far afield, so we will take a different approach more suitable for noncommutative geometry. The definition of spin^c has been shown by Plymen, [74], to be equivalent to the following straightforward characterisation in terms of Clifford algebras.

Definition 3.2 ([74]). Let (M, g) be an oriented Riemannian manifold. Then we say that (M, g) is spin^c if there exists a complex vector bundle $S \to M$ such that for all $x \in M$ the vector space S_x is an irreducible representation space for $\mathbb{C}\mathrm{liff}_x(M, g)$.

A spin^c structure on a Riemannian manifold (M, g) is then the choice of an orientation and irreducible representation bundle S of $\mathbb{C}\mathrm{liff}(M, g)$. The bundle S is called a (complex) spinor bundle.

When M has at least one spin^c structure S, we can build a new operator called the Dirac operator. Choose a Hermitian form $(\cdot\,|\,\cdot)$, and let $\nabla\colon \Gamma(S) \to \Gamma(T^*M \otimes S)$ be any connection compatible with $(\cdot\,|\,\cdot)$, which satisfies

$$\nabla^E(c(\varphi)\sigma) = c(\nabla^{T^*M}\varphi)\sigma + c(\varphi)\nabla^E\sigma \tag{3.1}$$

for all $\sigma \in \Gamma(E)$ and $\varphi \in \Gamma(T^*M)$. Then we compose the connection with the Clifford action

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S).$$

The resulting operator $\mathcal{D} = c \circ \nabla\colon \Gamma(S) \to \Gamma(S)$ is the Dirac operator on the (complex) spinor bundle S.

Using the more geometric definitions in terms of principal bundles, one can obtain a more canonical Dirac operator by taking ∇ to be a 'lift' of the Levi-Civita connection to S. This guarantees the existence of a connection satisfying eq. (3.1).

By considering connections on a $\mathbb{C}\mathrm{liff}(M)$ module, we see that we have a general construction of a Dirac operator on any such module. The following lemma describes these modules.

Lemma 3.3 (see [10]). *If M is a* spin^c *manifold, then every* $\mathbb{C}\mathrm{liff}(M)$ *module is of the form $S \otimes W$ where S is an irreducible* $\mathbb{C}\mathrm{liff}(M)$ *module and W is a complex vector bundle.*

Thus, on a spin^c manifold we can describe every 'Dirac type' operator as a twisted version of the Dirac operator of an irreducible Clifford module. In fact, using Poincaré duality in K-theory, one can show that up to homotopy and change of the order of the operator, every elliptic operator on a compact spin^c manifold is a twisted Dirac operator. See [54], Theorem 3.51.

One very important difference between the Dirac operator of a spin^c structure and the Hodge–de Rham operator must be mentioned. Recall the complex volume form defined in terms of a local orthonormal basis of the cotangent bundle by $\omega_{\mathbb{C}} = i^{[(n+1)/2]}e_1 \cdot e_2 \dots e_n$. This element of the Clifford algebra is globally parallel, meaning

$\nabla\omega_{\mathbb{C}} = 0$, where ∇ is the Levi-Civita connection. Since for any differential one-form φ we have

$$\varphi\omega_{\mathbb{C}} = (-1)^{n-1}\omega_{\mathbb{C}}\varphi,$$

we have the following two situations.

When n is odd, $\omega_{\mathbb{C}}$ is central with eigenvalues ± 1. Since the Clifford algebra is (pointwise) $M_{2^{(n-1)/2}}(\mathbb{C}) \oplus M_{2^{(n-1)/2}}(\mathbb{C})$, we can take $\omega_{\mathbb{C}} = 1 \oplus -1$. An irreducible representation of Cliff(M) then corresponds (pointwise) to a representation of one of the two matrix subalgebras. Without loss of generality we choose the representation with $\omega_{\mathbb{C}} = 1$. In this case the spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is ungraded, or odd.

When n is even, the ± 1 eigenspaces of $\omega_{\mathbb{C}}$ provide a global splitting of $S = S^+ \oplus S^-$. The Clifford action of a one-form maps S^+ to S^- and vice versa. Hence

$$\omega_{\mathbb{C}}\mathcal{D} = -\mathcal{D}\omega_{\mathbb{C}}.$$

As the Clifford algebra is (pointwise) a single matrix algebra, we get a representation of the whole Clifford algebra. The resulting spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is graded by the action of $\omega_{\mathbb{C}}$, and we get an even spectral triple.

Thus the Dirac operator of a spin^c *structure gives an even spectral triple if and only if the dimension of M is even. The same remains true if we twist the Dirac operator by any vector bundle.*

One can define a spin structure in terms of representations of real Clifford algebras. This is not quite a straightforward generalisation, but does go through: see [65], [51], [74]. Thus one can also talk about the Dirac operator of a spin structure. In many ways the spin case is easier, and certainly is so from the differential geometry point of view; see [65], Appendix D.

3.4 Relationship to the Atiyah–Singer index theorem. The Hodge–de Rham operator has index equal to the Euler characteristic of the manifold. In dimension 2, the Gauss–Bonnet theorem asserts that

$$\chi(M) = \frac{1}{2\pi}\int_M r\, d\mathrm{vol},$$

where r is the scalar curvature of M. This is a remarkable formula because it allows us to compute a topological quantity, $\chi(M)$, using geometric quantities. More blatantly, it says that by doing explicitly computable calculus operations, we can compute this topological invariant. The answer does not depend on which coordinates or metric we choose to compute with.

The Atiyah–Singer index theorem generalises[2] this theorem to any elliptic operator on a compact oriented n-dimensional manifold. Specifically, it says:

[2] There were several distinct motivations for Atiyah and Singer's work, from the integrality of the $\hat{A}$-genus to the Riemann–Roch theorem. The main point is that a range of different theorems became special instances of the Atiyah–Singer index theorem. See [95] for more information.

- take a first order elliptic operator $D\colon \Gamma(E) \to \Gamma(F)$ between sections of (complex) vector bundles $E,\ F \to M$;
- then there is a sum of even differential forms $\omega(D) = \omega_0(D) + \cdots + \omega_{2[n/2]}(D)$ so that
$$\mathrm{index}(D) = \int_M \omega_{2[n/2]}(D).$$
In particular, if n is odd the index is zero.

However, more is true. If we take a vector bundle W and twist everything to get $D \otimes_\nabla \mathrm{id}_W$ then
$$\mathrm{index}(D \otimes_\nabla \mathrm{id}_W) = \int_M \omega(D) \wedge \mathrm{Ch}(W).$$
Here $\mathrm{Ch}(W)$ is the Chern character of W defined by $\mathrm{Ch}(W) = \mathrm{trace}(e^{-\nabla^2})$ where ∇ is any connection on W and trace is a pointwise matrix trace. We will give another description of the Chern character of vector bundles later.

Thus the Atiyah–Singer index theorem allows us to not only compute the index of D, but all twisted versions of D also. This means that D is a machine for turning vector bundles into integers via
$$W \mapsto \mathrm{index}(D \otimes_\nabla \mathrm{id}_W) \in \mathbb{Z}.$$

Noncommutative geometry and the local index formula are about extending this kind of picture to the noncommutative world. Exchanging vector bundles over a space for projections ‘over’ an algebra enables pairings of spectral triples (differential operators) with K-theory. However there is also odd K-theory, described in terms of unitaries ‘over’ an algebra, and one may ask what the Atiyah–Singer theorem says in this case.

Actually, it is ‘the same’ in odd dimensions, the difference being that the Chern character of a unitary u (see Section 6.1) has only odd degree differential form components. Thus $\omega(D) \wedge \mathrm{Ch}(u)$ is an odd form, and so there can be forms of degree $\dim M$ to integrate over M. Also, it is not so straightforward to write down a differential operator whose index we are computing (it can be done) and we should think of the odd case in terms of generalised Toeplitz operators. That is, the odd index theorem should be thought of as computing $\mathrm{index}(PuP)$, where $P = \chi_{[0,\infty)}(\mathcal{D})$ is the non-negative spectral projection of $\mathcal{D}$.

Example 9. Hodge–de Rham. Here the sum of differential forms ω is given in [50], p. 175, and the term in top degree is given by $(2\pi)^{-n/2}\mathrm{Pf}(-R)$, where Pf is short for the Pfaffian of an antisymmetric matrix, and R is the curvature of the Levi-Civita connection. The Pfaffian satisfies $\mathrm{Pf}(A)^2 = \det(A)$, and changes sign if the orientation changes.

In the case where $\dim M = 2$, $\mathrm{Pf}(-R) = r$, the scalar curvature, and so
$$\mathrm{index}((d + d^*)^+) = \chi(M) = \frac{1}{2\pi}\int_M r\, d\mathrm{vol},$$
and we recover the classical Gauss–Bonnet theorem.

Example 10. For the signature operator,

$$\operatorname{index}((d + d^* \otimes_\nabla \mathrm{id}_E)^+) = (\pi i)^{-n/2} \int_M L(M) \wedge \mathrm{Ch}(E),$$

where the L-genus is $L = 1 + \frac{1}{24}\mathrm{Tr}(R^2) + \cdots$.

Example 11. For the spin Dirac operator,

$$\operatorname{index}((\mathcal{D} \otimes_\nabla \mathrm{id}_E)^+) = (2\pi i)^{-n/2} \int_M \hat{A}(M) \wedge \mathrm{Ch}(E),$$

where $\hat{A}$ is called the 'A-roof' or 'A-hat' class. So

$$\operatorname{index}(\mathcal{D}^+) = (2\pi i)^{-n/2} \int_M \hat{A}(M).$$

Since $\hat{A}(M)$ is defined in terms of the curvature, the index is independent of the spin structure. The right-hand side is always a rational number, and if it is not an integer, then the manifold has no spin structure.

These examples, and a discussion of the Atiyah–Singer index theorem, are presented in detail in [10], [48], [65].

A natural question at this point is whether the schematic

$$\text{index} \equiv \text{integral of differential forms}$$

has any sensible generalisation for noncommutative algebras. In a very real sense, cyclic homology is the generalisation of de Rham cohomology, and obtaining formulae for the index in terms of cyclic homology and cohomology is analogous to integrating differential forms to compute the index. We will take this up later.

3.5 The noncommutative torus and isospectral deformations. One of the nicest and most thoroughly studied spectral triples is defined for the irrational rotation algebra (and its higher dimensional relatives). The spectral triple defined below satisfies every proposed condition intended to characterise what we mean by a noncommutative manifold, [36].

The noncommutative torus is the universal unital C*-algebra A_θ generated by two unitaries U, V subject to the commutation relations

$$UV = e^{-2\pi i\theta} VU, \quad \theta \in [0, 1).$$

For $\theta = 0$ this is clearly the algebra of continuous functions on the torus. For θ rational, we obtain an algebra Morita equivalent to the functions on the torus. We will be interested in the case where θ is irrational. In this case, A_θ is simple.

There are two other descriptions of A_θ, which we mention for those interested in pursuing the noncommutative geometry of foliations and crossed products. The first is (up to stable isomorphism) as the C*-algebra associated to the Kronecker foliation of the (ordinary) 2-torus given by the differential equation

$$dy = \theta dx.$$

Spectral triples can be constructed for more general foliations also; see [60] and references therein.

The other description of A_θ is as a crossed product algebra (see [94] for a thorough introduction to this topic). For this description we have

$$A_\theta = C(S^1) \times_{R_\theta} \mathbb{Z}.$$

Here R_θ is the automorphism of $C(S^1)$ induced by a rotation of the circle by $2\pi\theta$. So for $g \in C(S^1)$ and $z \in S^1$ we have $(R_\theta(g))(z) = g(e^{2\pi i\theta} z)$. The connection to our generators U and V is as follows. The unitary $U \in C(S^1)$ is the generator of functions on the circle, the function $z \mapsto z$. The unitary $V \in A_\theta$ is the image of $1 \in \mathbb{Z}$ in the crossed product algebra. The unitary V implements the automorphism R_θ of $C(S^1)$. So, just considering the action of R_θ on U, we have

$$R_\theta(U) = VUV^* = e^{2\pi i\theta} U.$$

There is to our knowledge no general construction of spectral triples for crossed products $A \rtimes \Gamma$ given a discrete group Γ and spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Some special cases can be found in the literature, and we point out the recent paper [8] which analyses spectral triples for crossed products by $\mathbb{Z}$ from a metric point of view, in the sense of Section 2.

In order to build a spectral triple encoding geometry on the noncommutative torus, we need a smooth algebra, an unbounded operator and a Hilbert space. We begin with the algebra. Let

$$\mathcal{A}_\theta = \Big\{ \sum_{n,m\in\mathbb{Z}} c_{nm} U^n V^m : |c_{nm}|(|n| + |m|)^q \text{ is a bounded double sequence for all } q \in \mathbb{N} \Big\}.$$

Fourier theory on the ordinary torus suggests viewing this algebra as the smooth functions on the noncommutative torus. It is not much work to see that $\mathcal{A}_\theta$ is indeed a Fréchet pre-C*-algebra (see Section 5.1).

Next we require a Hilbert space. Recall that for θ irrational, A_θ has a unique faithful normalised trace ϕ given (on polynomials in the generators) by

$$\phi(a) = \phi\Big(\sum c_{ij} U^i V^j\Big) = c_{00}.$$

If we set $\langle a, b\rangle = \phi(b^* a)$, then $\langle \cdot, \cdot \rangle$ is an inner product and this makes A_θ a pre-Hilbert space. Completing with respect to the topology given by the inner product gives us a

Hilbert space which we write as $L^2(A_\theta, \phi)$. The algebra A_θ acts on $L^2(A_\theta, \phi)$ in the obvious way as multiplication operators. We set

$$\mathcal{H} = L^2(A_\theta, \phi) \oplus L^2(A_\theta, \phi).$$

This doubling up of the Hilbert space is motivated by the dimension of spinor bundles on the ordinary torus, or equivalently, the dimension of irreducible representations of $\mathbb{C}\mathrm{liff}(\mathbb{C}^2)$. Thus, loosely speaking, we are aiming to build a Dirac triple rather than a Hodge–de Rham triple.

In order to specify a Dirac operator for our triple, we need to look at how geometric data are encoded for classical tori. The problem is that any quadrilateral with opposite sides identified gives rise to a *geometric* object which is homeomorphic to the torus. To specify the extra geometric content given by our original quadrilateral, we embed it in the first quadrant of the complex plane with one vertex at the origin and another at $1 \in \mathbb{R}$ (we could scale the geometry up by putting one corner at $r \in \mathbb{R}$, but this is more or less irrelevant). The resulting geometry is then specified by the ratio of the edge lengths as complex numbers, or, with our description, a single complex number τ which is the coordinate of the other independent vertex. In particular, $\mathrm{im}(\tau) > 0$. The usual 'square' torus corresponds to the choice $\tau = i$.

With this in mind we define two derivations on A_θ by

$$\begin{aligned} &\delta_1(U) = 2\pi i U, \quad \delta_1(V) = 0, \\ &\delta_2(U) = 0, \qquad\quad\; \delta_2(V) = 2\pi i V. \end{aligned}$$

We then find (using $UU^* = 1$ etc and the Leibniz rule) that $\delta_1(U^*) = -2\pi i U^*$, $\delta_2(V^*) = -2\pi i V^*$, $\delta(U^n) = n2\pi i U^n$ and so on. These rules correspond to the derivatives of exponentials generating the functions on a torus. Using these derivations and a choice of τ with $\mathrm{im}(\tau) > 0$ we define

$$\mathcal{D} = \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix}.$$

Lastly, we set

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that we have the following heuristic. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2\mathrm{Re}(\tau) & 0 \\ 0 & -2\mathrm{Re}(\tau) \end{pmatrix},$$

it appears we are working with the Clifford algebra of the inner product

$$g = \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & |\tau|^2 \end{pmatrix}.$$

Again, if $\tau = i$, we are reduced to the usual Euclidean inner product.

Hard question: Can we encode a nonconstant metric using this heuristic?

We claim $(\mathcal{A}_\theta, \mathcal{H}, \mathcal{D})$ defines an even spectral triple with grading γ for each such choice of τ. First we must show that for all $a \in \mathcal{A}_\theta$ we have $[\mathcal{D}, a]$ bounded. So let $a = \sum c_{nm} U^n V^m \in \mathcal{A}_\theta$. Then

$$\begin{aligned}\mathcal{D}a - a\mathcal{D} &= \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar\tau\delta_2 & 0 \end{pmatrix} \begin{pmatrix} \sum c_{nm} U^n V^m & 0 \\ 0 & \sum c_{nm} U^n V^m \end{pmatrix} \\ &\quad - \begin{pmatrix} \sum c_{nm} U^n V^m & 0 \\ 0 & \sum c_{nm} U^n V^m \end{pmatrix} \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar\tau\delta_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\pi i \sum c_{nm} U^n V^m (n + m\tau) \\ -2\pi i \sum c_{nm} U^n V^m (n + m\bar\tau) & 0 \end{pmatrix},\end{aligned}$$

and as $|c_{nm}|$ is 'Schwartz class', this converges in norm and so is bounded.

Next we must show that $\mathcal{D}$ has compact resolvent. As this is equivalent to $\mathcal{D}$ having only eigenvalues of finite multiplicity (which must go to infinity so that $\mathcal{D}$ is unbounded) we will prove this instead. We begin by looking at $\mathcal{D}^2$,

$$\mathcal{D}^2 = \begin{pmatrix} -\delta_1^2 - |\tau|^2\delta_2^2 - \bar\tau\delta_1\delta_2 - \tau\delta_2\delta_1 & 0 \\ 0 & -\delta_1^2 - |\tau|^2\delta_2^2 - \tau\delta_1\delta_2 - \bar\tau\delta_2\delta_1 \end{pmatrix}.$$

Applying this to the monomial $U^n V^m \oplus U^n V^m$ gives

$$\begin{aligned} D^2 \begin{pmatrix} U^n V^m \\ U^n V^m \end{pmatrix} &= (2\pi)^2 (n^2 + |\tau|^2 m^2 + nm(\tau + \bar\tau)) \begin{pmatrix} U^n V^m \\ U^n V^m \end{pmatrix} \\ &= (2\pi)^2 |n + \tau m|^2 \begin{pmatrix} U^n V^m \\ U^n V^m \end{pmatrix}. \end{aligned}$$

This shows that all of these monomials are eigenvectors of $\mathcal{D}^2$. Note that

$$\phi(V^{-l}U^{-k}U^n V^m) = \phi(V^{-l}U^{n-k}V^m) = \phi(e^{-2\pi i l\theta(n-k)}U^{n-k}V^{m-l}) = \delta_{n,k}\delta_{m,l},$$

so that the monomials $U^n V^m$ form an orthonormal basis of $L^2(A_\theta, \phi)$ (they clearly span). As $\mathcal{D}^2$ preserves the splitting of $\mathcal{H}$, we see that these are all the eigenvalues of $\mathcal{D}^2$ and that they give the whole spectrum of $\mathcal{D}^2$. Also note in passing that

$$\ker \mathcal{D}^2 = \mathrm{span}_{\mathbb{C}}\{1\} \oplus \mathrm{span}_{\mathbb{C}}\{1\} = \mathbb{C} \oplus \mathbb{C}.$$

Our results so far are actually enough to conclude, but let us make the eigenvalues and eigenvectors of $\mathcal{D}$ explicit.

The eigenvalues of $\mathcal{D}$ are given by the square roots of the eigenvalues of $\mathcal{D}^2$, and so are

$$\pm 2\pi|n + \tau m|, \quad n, m \in \mathbb{Z}.$$

The corresponding eigenvectors are

$$\text{positive sign} \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^n V^m \\ U^n V^m \end{pmatrix}, \quad \text{negative sign} \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^n V^m \\ -U^n V^m \end{pmatrix}.$$

The multiplicity of these eigenvalues depends on the value of τ, but is always finite. Thus $\mathcal{D}$ has compact resolvent, and for any choice of τ with $\mathrm{im}(\tau) > 0$ we have an even spectral triple.

Using the noncommutative torus we can construct other 'noncommutative manifolds', which are isospectral deformations of classical manifolds. The word 'isospectral' refers to the fact that while the new spectral triple is in a suitable sense a deformation of the original, the spectrum of the Dirac operator remains unchanged.

Part of the interest here is that for a classical geometric space (i.e., a manifold or domain in $\mathbb{R}^n$) the spectrum of the Laplace operator comes very close to characterising the space plus metric up to isometry. An important point is that there are non-isometric spaces whose Laplacians have the same spectrum, but nevertheless the spectrum of the Laplacian remains a strong invariant.

This goes back to the famous paper 'Can one hear the shape of a drum?', [58], and lives on in the subject known as spectral geometry. A good overview can be found in [50], while an interesting modern contribution is [2]. See also references in http://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum.

Thus, making an isospectral deformation, but making the algebra become noncommutative, should result in an 'almost' complete characterisation of the resulting noncommutative space up to isometry.

Let (M, g) be a compact Riemannian spin manifold with an isometric action of $\mathbb{T}^2$ (we can do this with higher dimensional noncommutative tori also). Then we define $C^\infty(M_\theta)$ to be the fixed point for the diagonal action of $\mathbb{T}^2$ on $C^\infty(M) \otimes A_\theta$. This is similar to gluing in a noncommutative torus in place of each torus orbit in M. The algebra $C^\infty(M_\theta)$ is noncommutative.

The same kind of procedure (though somewhat more subtle) applied to the Hilbert space of spinors allows one to take $(C^\infty(M), L^2(S), \mathcal{D})$ and produce $(C^\infty(M_\theta), L^2(S_\theta), \mathcal{D})$, where $\mathcal{D}$ is essentially the same (Dirac) operator in both triples, and certainly has the same spectrum. The interested reader can look at the papers by Alain Connes and Michel Dubois-Violette, [37].

4 K-theory, K-homology and the index pairing

4.1 K-theory. This section is the briefest of overviews of K-theory for C*-algebras. If the discussion here is unfamiliar, try [92], [84], [54]. We will follow [54]. The reasons we spend much more time on K-homology than on K-theory are that there are many more texts on K-theory, and because spectral triples represent K-homology classes.

Definition 4.1. Given a unital C*-algebra A, we denote by $K_0(A)$ the abelian group with one generator $[p]$ for each projection p in each matrix algebra $M_n(A)$, $n = 1, 2, \ldots$, and the following relations:

a) if $p, q \in M_n(A)$ and p, q are joined by a norm continuous path of projections in $M_n(A)$ then $[p] = [q]$;

b) $[0] = 0$ for any square matrix of zeroes;

c) $[p] + [q] = [p \oplus q]$ for any $[p]$, $[q]$.

In a), we say that p and q are homotopic. If $p \in M_n(A)$ we say that p is a projection over A. Projections in matrix algebras $M_n(A)$ and $M_m(A)$, $m \neq n$, can be compared by using the fact that these matrix algebras form an increasing union in the obvious way (that is, $M_n(A) \subset M_{n+1}(A)$).

The group structure arises using the Grothendieck construction: every element of $K_0(A)$ can be written as a formal difference $[p] - [q]$, and two elements $[p] - [q]$ and $[p'] - [q']$ are equal if and only if there is a projection r such that

$$p \oplus q' \oplus r \text{ is homotopic to } p' \oplus q \oplus r.$$

The assignment $A \to K_0(A)$ is a covariant functor from C*-algebras to abelian groups. If $\phi\colon A \to B$ is a $*$-homomorphism, then applying ϕ element by element to the matrix $p \in M_n(A)$ gives a projection $\phi(p) \in M_n(B)$. This yields a group homomorphism $\phi_*\colon K_0(A) \to K_0(B)$.

Exercise. Show that two projections in $M_n(\mathbb{C})$ are homotopic if and only if they have the same rank, and that $[p] \to \text{rank}(p)$ is an isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

Example 12 (*Important*). If $A = C(X)$, where X is a compact Hausdorff space, we find that $K_0(A) = K^0(X)$, where $K^0(X)$ is the topological K-theory defined by vector bundles.

This is because if $E \to X$ is a complex vector bundle, there is a projection $p \in M_N(C(X))$ for some N such that $\Gamma(X, E) \cong pC(X)^N$ as a $C(X)$-module. Similarly, any $C(X)$-module of the form $pC(X)^N$ is the space of sections of a vector bundle. This is the Serre–Swan theorem, [88], and we recommend the book [3] as a wonderful place to discover this and other aspects of topological K-theory.

This exchange between projections and vector bundles is one of the many important instances of exchanging topological information for algebraic information, with the Gel´fand Theory (exchanging abelian C*-algebras and Hausdorff spaces) being one of the main motivations of noncommutative geometry.

Definition 4.2. Given a unital C*-algebra A, we denote by $K_1(A)$ the abelian group with one generator $[u]$ for each unitary u in each matrix algebra $M_n(A)$, $n = 1, 2, \ldots$, and the following relations:

a) if $u, v \in M_n(A)$ and u, v are joined by a norm continuous path of unitaries in $M_n(A)$ then $[u] = [v]$;

b) $[1] = 0$ for any square identity matrix;

c) $[u] + [v] = [u \oplus v]$ for any $[u]$, $[v]$.

Note that unitaries in matrix algebras of different dimensions can be compared by using the trick of sending u to

$$\begin{pmatrix} u & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Exercise. Let $\sim$ denote the relation of path-connectedness through unitaries. Let $u,\ v \in A$ be unitary. Prove that in $M_2(A)$ we have

$$u \oplus 1 \sim 1 \oplus u, \quad u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \quad u \oplus u^* \sim 1 \oplus 1.$$

Hint: consider the rotation matrix

$$R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

From the exercise we see that $[u]+[u^*] = [u \oplus u^*] = [1] = 0$, and so $-[u] = [u^*]$.

Exercise. Show that $K_1(\mathbb{C}) = 0$.

This section on K-theory could be made arbitrarily long, but it is not the main focus of these notes, and so we leave K-theory for now with the warning that here we have seen the definitions and nothing more.

4.2 Fredholm modules and K-homology. One of the central reasons that the techniques employed by Atiyah and Singer to compute the index of elliptic differential operators on manifolds continues to work for noncommutative spaces is the way K-theory enters the proof. Essentially, K-theory, and the dual theory K-homology, make perfectly good sense for C*-algebras, commutative or not. The best reference for K-homology is the book [54]. The origins of the ideas can be seen in [4], and were fully developed for the commutative case in [14] while the modern form is developed in [59]. While we were very brief with K-theory, we will spend a little longer on K-homology as it is much closer to spectral triples. Indeed, spectral triples are 'just' nice representatives of classes in K-homology. For example, cohomology of manifolds is studied using tools of calculus by introducing differential forms. Spectral triples play an almost exactly analogous role.

Definition 4.3. Let A be a separable C*-algebra. A Fredholm module over A is given by a Hilbert space $\mathcal{H}$, a $*$-representation $\rho\colon A \to \mathcal{B}(\mathcal{H})$ and an operator $F : \mathcal{H} \to \mathcal{H}$ such that, for all $a \in A$,

$$(F^2 - 1)\rho(a), \quad (F - F^*)\rho(a), \quad [F, \rho(a)] := F\rho(a) - \rho(a)F$$

are all compact operators. We say that $(\rho, \mathcal{H}, F)$ is even (or graded) if there is an operator $\gamma\colon \mathcal{H} \to \mathcal{H}$ such that $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma F + F\gamma = 0$ and $\gamma\rho(a) = \rho(a)\gamma$ for all $a \in A$. Otherwise we call $(\rho, \mathcal{H}, F)$ odd.

We will usually consider algebras A which are unital and for which $\rho(1_A) = \mathrm{id}_{\mathcal{H}}$, and this simplifies the first two conditions on F: $F^2 - 1$ and $F - F^*$ are compact. In this case we have the following descriptions.

An odd Fredholm module is given by a (unital) representation ρ on $\mathcal{H}$ and an operator $F = 2P - 1 + K$, where K is compact and P is a projection commuting with $\rho(A)$ modulo compact operators

An even Fredholm module is given by a pair of representations ρ_+, ρ_- on Hilbert spaces $\mathcal{H}_+, \mathcal{H}_-$ respectively, and

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \rho = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}, \tag{4.1}$$

with $F_- = (F_+)^* + K$ with K compact. The conditions defining the Fredholm module tell us that $F_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is a Fredholm operator.

Example 13. Let $\rho\colon \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be the unique unital representation. Then an ungraded Fredholm module is given by an essentially self-adjoint Fredholm operator F. Likewise, a graded Fredholm module is given by an essentially self-adjoint Fredholm operator of the form (4.1).

For an even Fredholm module, we denote by $\mathrm{index}(\rho, \mathcal{H}, F)$ the index of $F_+ : \mathcal{H}_1 \to \mathcal{H}_2$.

Example 14. Let $\mathcal{H} = L^2(S^1)$ and represent $C(S^1)$ on $\mathcal{H}$ as multiplication operators. So for $f \in C(S^1)$ and $\xi \in L^2(S^1)$, $(f\xi)(x) = f(x)\xi(x)$ for $x \in S^1$.

Let $P \in \mathcal{B}(\mathcal{H})$ be the projection onto $\overline{\mathrm{span}}\{z^k : k \geq 0\}$. Since P is a projection, the operator $F = 2P - 1$ is self-adjoint and has square one. Thus, to check that $(\mathcal{H}, F)$ (along with the multiplication representation) is an odd Fredholm module for $C(S^1)$, we just need to check that $[F, f]$ is compact for all $f \in C(S^1)$.

First, every $f \in C(S^1)$ is a norm (uniform) limit of trigonometric polynomials (Stone–Weierstrass) and so the norm limit of finite sums of z^k, $k \in \mathbb{Z}$, where $z\colon S^1 \to \mathbb{C}$ is the identity function.

Hence it suffices to show that $[F, z^k]$ is compact for each k, and so it is enough to show that $[P, z^k]$ is compact. Let $\xi \in \mathcal{H}$ so $\xi = \sum_{n\in\mathbb{Z}} c_n z^n$ (this sum converges in the Hilbert space norm). Then

$$Pz^k\xi = P\sum_{n\in\mathbb{Z}} c_n z^{n+k} = \sum_{n\geq -k} c_n z^{n+k},$$

while

$$z^k P\xi = \sum_{n\geq 0} c_n z^{n+k}.$$

The difference is

$$[P, z^k]\xi = \begin{cases} \sum_{n=-k}^{0} c_n z^{n+k}, & k \geq 0, \\ \sum_{n=0}^{-k} c_n z^{n+k}, & k \leq 0, \end{cases}$$

Hence $[P, z^k]$ is a rank k operator, and so compact.

The operators $T_f := PfP : P\mathcal{H} \to P\mathcal{H}$, $f \in C(S^1)$, are called Toeplitz operators. One can show that

$$T_f^* = T_{\bar{f}} \quad \text{and} \quad T_f T_g = T_{fg} + K,$$

where K is a compact operator. Composing with the quotient map $\pi : \mathcal{B}(P\mathcal{H}) \to \mathcal{Q}(P\mathcal{H})$ we get a $*$-homomorphism $C(S^1) \to \mathcal{Q}(P\mathcal{H})$ which is *faithful*!; see [54] for instance. Hence we get an extension (short exact sequence)

$$0 \to \mathcal{K}(P\mathcal{H}) \to T \to C(S^1) \to 0,$$

where T is the algebra generated by the T_f, $f \in C(S^1)$. This is called the Toeplitz extension.

Exercise. What is the relationship between the Fredholm module for $C(S^1)$ in Example 14 and the spectral triple in Example 4? *Hint*: Look at the next example.

Example 15. Let $\mathcal{H} = L^2(\bigwedge^* M, g)$ and let $C^\infty(M)$ act as multiplication operators. If $\mathcal{D} = d + d^*$, then we know that $(1 + \mathcal{D}^2)^{-1/2}$ is compact, and $F_\mathcal{D} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ is bounded (by the functional calculus). Now we compute

$$F_\mathcal{D}^2 = \mathcal{D}^2(1 + \mathcal{D}^2)^{-1} = 1 - (1 + \mathcal{D}^2)^{-1},$$

which is the identity modulo compacts. Since $F_\mathcal{D}$ is self-adjoint and anticommutes with the grading γ of differential forms by degree, we need only check that $[F_\mathcal{D}, f]$ is compact for all $f \in C^\infty(M)$. So

$$[F_\mathcal{D}, f] = [\mathcal{D}, f](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, f](1 + \mathcal{D}^2)^{-1/2}.$$

Since $[\mathcal{D}, f]$ is Clifford multiplication by df, the first term is compact. Likewise, the second term will be compact if we can see that $[(1 + \mathcal{D}^2)^{1/2}, f]$ is bounded. But the symbol of f is $f\,\mathrm{id}$, so the order of the commutator is $1 + 0 - 1 = 0$, and so we have a bounded pseudodifferential operator. Hence we get an even Fredholm module for the algebra $C^\infty(M)$.

Exercise. Does the Fredholm module of Example 15 extend to a Fredholm module for $C(M)$, the C*-algebra of continuous functions? Why?

As the example suggests, every spectral triple defines a Fredholm module. For simplicity, we give a proof that relies on an extra assumption, though this can be removed (see [17]).

Proposition 4.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple such that $[|\mathcal{D}|, a]$ is bounded for all $a \in \mathcal{A}$, where $|\mathcal{D}| = \sqrt{\mathcal{D}^2}$ is the absolute value. Define $F_\mathcal{D} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. Then $(\mathcal{H}, F_\mathcal{D})$ is a Fredholm module for the C^*-algebra $A := \overline{\mathcal{A}}$.*

Proof. For $a \in \mathcal{A}$ we have

$$[F_{\mathcal{D}}, a] = [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} + \mathcal{D}[(1 + \mathcal{D}^2)^{-1/2}, a]$$
$$= [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2}.$$

This is a compact operator. If $\{a_k\}_{k\geq 0} \subset \mathcal{A}$ is a sequence converging in (operator) norm, then

$$\|[F_{\mathcal{D}}, a_k - a_m]\| \leq 2\|F_{\mathcal{D}}\| \, \|a_k - a_m\| \leq 2\|a_k - a_m\| \to 0.$$

Hence if $a = \lim_k a_k$ with $a_k \in \mathcal{A}$ and convergence in norm,

$$[F_{\mathcal{D}}, a] = \lim_k [F_{\mathcal{D}}, a_k]$$

and this is a limit of compact operators, and so compact. □

Definition 4.5. Let $(\rho, \mathcal{H}, F)$ be a Fredholm module, and suppose that $U : \mathcal{H}' \to \mathcal{H}$ is a unitary. Then $(U^*\rho U, \mathcal{H}', U^*FU)$ is also a Fredholm module (with grading $U^*\gamma U$ if γ is a grading of $(\rho, \mathcal{H}, F)$) and we say that it is unitarily equivalent to $(\rho, \mathcal{H}, F)$.

Definition 4.6. Let $(\rho, \mathcal{H}, F_t)$ be a family of Fredholm modules parameterised by $t \in [0, 1]$ with $\rho, \mathcal{H}$ constant. If the function $t \to F_t$ is norm continuous, we call this family an operator homotopy between $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$, and say that these two Fredholm modules are operator homotopic.

If $(\rho_1, \mathcal{H}_1, F_1)$ and $(\rho_2, \mathcal{H}_2, F_2)$ are Fredholm modules over the same algebra A, then $(\rho_1 \oplus \rho_2, \mathcal{H}_1 \oplus \mathcal{H}_2, F_1 \oplus F_2)$ is a Fredholm module over A, called the direct sum.

Definition 4.7. Let $p = 0, 1$. The K-homology group $K^p(A)$ is the abelian group with one generator $[x]$ for each unitary equivalence class of Fredholm modules (even or graded if $p = 0$, and odd or ungraded for $p = 1$) with the following relations:

1) If x_0 and x_1 are operator homotopic Fredholm modules (both even or both odd), then $[x_0] = [x_1]$ in $K^p(A)$.
2) If x_0 and x_1 are two Fredholm modules (both even or both odd), then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $K^p(A)$.

The zero element is the class of the zero module, which is the zero Hilbert space, zero representation and naturally a zero operator. There are also other representatives of this class, which we require in order to be able to display inverses.

Definition 4.8. A Fredholm module $(\rho, \mathcal{H}, F)$ is called degenerate if $F = F^*$, $F^2 = 1$ and $[F, \rho(a)] = 0$ for all $a \in A$.

Exercise. The class of a degenerate module is zero in K-homology. *Hint*: Consider $\oplus^\infty(\rho, \mathcal{H}, F)$ and $(\rho, \mathcal{H}, F) \oplus \oplus^\infty(\rho, \mathcal{H}, F)$.

Lemma 4.9 ([54]). *If $x = (\rho, \mathcal{H}, F)$ is an odd Fredholm module, then the class of $-[x]$ is represented by the Fredholm module $(\rho, \mathcal{H}, -F)$. For an even Fredholm module $x = (\rho, \mathcal{H}, F, \gamma)$ the inverse is represented by $(\rho, \mathcal{H}, -F, -\gamma)$.*

Proof. We do the even case, by showing that

$$\left(\begin{pmatrix}\rho & 0\\ 0 & \rho\end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix}F & 0\\ 0 & -F\end{pmatrix}, \begin{pmatrix}\gamma & 0\\ 0 & -\gamma\end{pmatrix}\right)$$

is operator homotopic to the degenerate module

$$\left(\begin{pmatrix}\rho & 0\\ 0 & \rho\end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix}0 & \mathrm{id}_{\mathcal{H}}\\ \mathrm{id}_{\mathcal{H}} & 0\end{pmatrix}, \begin{pmatrix}\gamma & 0\\ 0 & -\gamma\end{pmatrix}\right).$$

We do this by displaying the homotopy

$$F_t = \begin{pmatrix}\cos(\pi t/2)F & \sin(\pi t/2)\mathrm{id}_{\mathcal{H}}\\ \sin(\pi t/2)\mathrm{id}_{\mathcal{H}} & -\cos(\pi t/2)F\end{pmatrix}.$$

We leave the details as an **Exercise**. □

Let $\psi : A \to B$ be a $*$-homomorphism, and $(\rho, \mathcal{H}, F)$ a Fredholm module over B. Then $(\rho \circ \psi, \mathcal{H}, F)$ is a Fredholm module over A. This allows us to define

$$\psi^* : K^*(B) \to K^*(A) \quad \text{by} \quad \psi^*[(\rho, \mathcal{H}, F)] = [(\rho \circ \psi, \mathcal{H}, F)]$$

and so K-homology is a contravariant functor from (separable) C*-algebras to abelian groups. We write $K^*(A) = K^0(A) \oplus K^1(A)$.

Being able to work modulo compact operators gives us plenty of freedom, and allows us to build a good (co)homology theory. Sometimes however, it is better to have 'nice' representatives of K-homology classes.

Lemma 4.10 ([54]). *Every K-homology class in $K^*(A)$ can be represented by a Fredholm module $(\rho, \mathcal{H}, F)$ with $F = F^*$ and $F^2 = 1$. Alternatively, we may suppose that $(\rho, \mathcal{H}, F)$ is nondegenerate in the sense that $\rho(A)\mathcal{H}$ is dense in $\mathcal{H}$. In general we cannot do both these things at the same time.*

We will call any Fredholm module with $F = F^*$ and $F^2 = 1$ a *normalised Fredholm module*. In [54], this is called an involutive Fredholm module. Usually, we will omit the representation, and refer to a Fredholm module $(\mathcal{H}, F)$ for a C*-algebra A.

Later we will give an explicit way of obtaining a normalised Fredholm module from a spectral triple. Here are simple methods for achieving the same end when we begin with a Fredholm module.

Exercise. Given a Fredholm module $(\rho, \mathcal{H}, F)$ for a C*-algebra A, show that $t \mapsto F_t = (1-t)F + \frac{t}{2}(F + F^*)$ is an operator homotopy, and F_1 is self-adjoint.

Exercise. Let $(\rho, \mathcal{H}, F)$ be a Fredholm module for a C*-algebra A with $F = F^*$. Show that $(\rho, \mathcal{H}, F)$ is equivalent (in K-homology) to

$$\left(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}, \tilde{F} = \begin{pmatrix}F & (1-F^2)^{1/2}\\ (1-F^2)^{1/2} & -F\end{pmatrix}\right),$$

and $\tilde{F}^2 = 1$.

4.3 The index pairing. The pairing between K-theory and K-homology is given in terms of the Fredholm index. As we need to handle matrix algebras over $\mathcal{A}$ we observe that if $(\mathcal{H}, F)$ is a Fredholm module for an algebra $\mathcal{A}$, then $(\mathcal{H}^k, F \otimes \mathrm{id}_k)$ is a Fredholm module for $M_k(\mathcal{A})$. If $(\mathcal{H}, F)$ is normalised so is $(\mathcal{H}^k, F \otimes \mathrm{id}_k)$. We leave this as an **Exercise**.

Let $(\mathcal{H}, F, \gamma)$ be an even Fredholm module for an algebra $\mathcal{A}$ and $p \in M_k(\mathcal{A})$ a projection. Then the pairing between $[p] \in K_0(\mathcal{A})$ and $[(\mathcal{H}, F, \gamma)] \in K^0(\mathcal{A})$ is given by

$$\langle [p], [(\mathcal{H}, F, \gamma)] \rangle := \mathrm{index}(p(F^+ \otimes \mathrm{id}_k)p \colon p\mathcal{H}^k \to p\mathcal{H}^k).$$

When $(\mathcal{H}, F)$ is an odd Fredholm module over $\mathcal{A}$, and $u \in M_k(\mathcal{A})$ is a unitary, the pairing between $[u] \in K_1(\mathcal{A})$ and $[(\mathcal{H}, F)] \in K^1(\mathcal{A})$ is given by

$$\langle [u], [(\mathcal{H}, F)] \rangle := \mathrm{index}(P_k u P_k - (1 - P_k) \colon \mathcal{H}^k \to \mathcal{H}^k),$$

where $P_k = \frac{1}{2}(1 + F) \otimes \mathrm{id}_k$. If P_k is actually a projection, then this formula simplifies to

$$\langle [u], [(\mathcal{H}, F)] \rangle := \mathrm{index}(P_k u P_k \colon P_k\mathcal{H}^k \to P_k\mathcal{H}^k).$$

The reason is that $1 - P_k$ maps $(1 - P_k)\mathcal{H}^k$ to itself, and is self-adjoint and so has zero index. When P_k is not a projection, we cannot just consider $P_k u P_k$, as it does not necessarily preserve any subspace. Adding $1 - P_k$ preserves the Fredholm property on the whole Hilbert space and does not change the index when P_k is a projection.

Since we can always find a normalised Fredholm module in any K-homology class, it is more useful to always think of $\mathrm{index}(P_k u P_k)$.

Exercise. With p, u, P_k, F as above, show that $p(F^+ \otimes \mathrm{id}_k)p$ and $P_k u P_k - (1 - P_k)$ are Fredholm operators.

Exercise. Show that the index pairing between a projection and an even Fredholm module depends only on the K classes. Similarly for the odd case.

When $P = (1 + F)/2$ is a projection (say when $F = F^*$ and $F^2 = 1$), operators such as $P_k u P_k$ are called generalised Toeplitz operators. The index of $P_k u P_k$ is also equal to spectral flow. This *spectral flow* has a rigorous analytic definition due to Phillips, [71], [72], which we won't pursue here. The original topological definition is due to Lusztig and Atiyah–Patodi–Singer, [6].

Example 16. Let $\mathcal{D}^+ : \Gamma(E) \to \Gamma(F)$ be a first order elliptic differential operator on the manifold M. Let $(C^\infty(M), L^2(E) \oplus L^2(F), \mathcal{D})$ be the even spectral triple with $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$, where $\mathcal{D}^- = (\mathcal{D}^+)^*$.

If W is another vector bundle, we can associate to it a projection $p \in M_N(C^\infty(M))$ so that

$$L^2(E \otimes W) = pL^2(E)^N,$$

and similarly for $F \otimes W$. Then $\mathcal{D} \otimes_\nabla \mathrm{id}_W = p(\mathcal{D} \otimes 1_N)p +$ order zero terms, and we see that

$$\mathrm{index}((\mathcal{D} \otimes_\nabla \mathrm{id}_W)^+) = \mathrm{index}(p(\mathcal{D}^+ \otimes 1_N)p).$$

Exercise. Prove the equalities in Example 16. *Hint*: If $\Gamma(E) = pC^\infty(M)^N$, then the composition

$$pC^\infty(M)^N \xrightarrow{i} C^\infty(M)^N \xrightarrow{d} \Gamma(T^*M)^N \xrightarrow{p} \Gamma(E \otimes T^*M)$$

is a connection.

From now on for notational convenience we will assume that any projection or unitary we want to pair with a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ lives in the algebra $\mathcal{A}$ rather than $M_k(\mathcal{A})$.

4.4 The index pairing for finitely summable Fredholm modules. In this section we briefly describe how we can compute the index pairing for (the class of) certain special Fredholm modules. Before stating the definition, we need a comment. There is a problem with C*-algebras as can be seen in the commutative case: if we want to start computing the index of $p\mathcal{D}p$ where $\mathcal{D}$ is a first order elliptic differential operator, we have to require that p is at least C^1. That means we cannot use just any representative of a K-theory class. Next there needs to be some additional constraint on the Fredholm module in order to obtain explicit formulae. For finite dimensional situations one uses 'finite summability' of commutators $[F, a]$, which we now proceed to define. Note that as usual our reference for operator ideals is [87].

Definition 4.11. For any $p \geq 1$, define the p-th Schatten ideal of the Hilbert space $\mathcal{H}$ to be

$$\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \text{trace}(|T|^p) < \infty\}, \quad |T| = \sqrt{T^*T}.$$

Remarks. 1) These ideals are all two-sided, but are not norm closed. As the compact operators are the only norm closed ideal of the bounded operators on Hilbert space (when the Hilbert space is separable), all the ideals $\mathcal{L}^p(\mathcal{H})$ have norm closure equal to the compact operators.

2) The ideal $\mathcal{L}^2(\mathcal{H})$ is called the Hilbert–Schmidt class, and is a Hilbert space for the inner product

$$\langle T, S\rangle = \text{trace}(S^*T) \quad \text{for all } T,\ S \in \mathcal{L}^2(\mathcal{H}).$$

In particular, it is complete for the norm $\|T\|_2 = \text{trace}(T^*T)^{1/2}$.

3) More generally, $\mathcal{L}^p(\mathcal{H})$ is complete for the norm

$$\|T\|_p := \text{trace}(|T|^p)^{\frac{1}{p}},\ T \in \mathcal{L}^p(\mathcal{H}).$$

Moreover, if $B, C \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^p(\mathcal{H})$, then $\|BTC\|_p \leq \|B\|\,\|T\|_p\,\|C\|$, where $\|\cdot\|$ is the usual operator norm.

4) Two useful (and immediate) facts are:

a) If $T \in \mathcal{L}^p(\mathcal{H})$ then $T^p \in \mathcal{L}^1(\mathcal{H})$.

b) If $T_i \in \mathcal{L}^{p_i}(\mathcal{H})$ then $T_1 \dots T_k \in \mathcal{L}^p(\mathcal{H})$, where

$$\frac{1}{p} = \sum_{j=1}^{k} \tfrac{1}{p_j}.$$

This last can be shown using the Hölder inequality. Note the analogy with the ℓ^p spaces of classical analysis. After this very brief summary of these operator ideals, we make the following key definition.

Definition 4.12 (Connes). Let $\mathcal{A}$ be a unital $*$-algebra. A (normalised) Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$ is $(p+1)$-summable, $p \in \mathbb{N}$, if for all $a \in \mathcal{A}$ we have

$$[F, a] \in \mathcal{L}^{p+1}(\mathcal{H}).$$

Example 17. When we construct the Hodge–de Rham Fredholm module, instead of starting with $d + d^*$, we can start with

$$\mathcal{D}_m = \begin{pmatrix} d + d^* & m \\ m & -(d + d^*) \end{pmatrix}, \quad m > 0,$$

acting on $\mathcal{H}_2 = L^2(\bigwedge^* M, g) \oplus L^2(\bigwedge^* M, g)$ with the grading $\gamma \oplus -\gamma$. The representation of $C^\infty(M)$ on $\mathcal{H}_2$ is as multiplication operators in the first copy, and by zero in the second. Since $\mathcal{D}_m$ is invertible, we are free to define $F_{\mathcal{D}_m} = \mathcal{D}_m|\mathcal{D}_m|^{-1}$. Again we obtain a Fredholm module, but this time $F_{\mathcal{D}_m}^2 = \mathrm{id}_{\mathcal{H}_2}$, and so we have a normalised Fredholm module. From Weyl's theorem, see Theorem 5.11, $|\mathcal{D}_m|^{-1}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \dots$ satisfying

$$\mu_n = C\, n^{-1/\dim M} + o(n^{-1/\dim M}).$$

From this and the standard commutator tricks, it is easy to see that $(\mathcal{H}_2, F_{\mathcal{D}_m})$ is $(\dim(M) + 1)$-summable.

For finitely summable normalised Fredholm modules we can define cyclic cocycles whose class in periodic cyclic cohomology is called the *Chern character*. We defer a detailed discussion of this cohomology theory to Section 6.1. We now give the definition of the Chern character and its main application.

Definition 4.13 (Connes). Let $(\mathcal{H}, F)$ be a $(p+1)$-summable normalised Fredholm module for the $*$-algebra $\mathcal{A}$. For any $n \geq p$ of the same parity as the Fredholm module we define cyclic cocycles by

$$\mathrm{Ch}_n(\mathcal{H}, F)(a_0, a_1 \dots, a_n) = \tfrac{\lambda_n}{2}\, \mathrm{trace}(\gamma[F, a_0][F, a_1] \dots [F, a_n]),$$

where γ is 1 if the module is odd, and the normalisation constants are

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2}\Gamma(n/2+1) & \text{(even)}, \\ \sqrt{2i}(-1)^{n(n-1)/2}\Gamma(n/2+1) & \text{(odd)}. \end{cases}$$

The Chern character $\mathrm{Ch}_*(\mathcal{H}, F)$ is the class of these cocycles in periodic cyclic cohomology.

There is a notational device for making the odd and even cases similar. For $T \in \mathcal{B}(\mathcal{H})$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$, define the 'conditional trace'

$$\mathrm{trace}'(T) = \tfrac{1}{2}\,\mathrm{trace}(F(FT + TF)).$$

Note that if $T \in \mathcal{L}^1(\mathcal{H})$ then $\mathrm{trace}'(T) = \mathrm{trace}(T)$. Then define

$$\mathrm{trace}_s(T) = \mathrm{trace}'(\gamma T).$$

Here $\gamma = \mathrm{id}_{\mathcal{H}}$ if n is odd. Then we can write

$$\mathrm{Ch}_n(\mathcal{H}, F, \gamma)(a_0, a_1, \dots, a_n) = \lambda_n\,\mathrm{trace}_s(a_0[F, a_1]\dots[F, a_n]).$$

Remark. This is the definition as given by Alain Connes, and the next theorem we take from [34] also, except for a minus sign in the odd case. This minus sign is a persistent nuisance in the literature, and we have addressed it by retaining the definition of the Chern character and introducing an additional minus sign in the pairing.

Theorem 4.14 (Connes). *Let $(\mathcal{H}, F)$ be a finitely summable normalised Fredholm module over $\mathcal{A}$. Then, for any $[e] \in K_0(\mathcal{A})$,*

$$\langle [e], [(\mathcal{H}, F)] \rangle = \mathrm{Ch}_*(\mathcal{H}, F)(e) := \frac{1}{(n/2)!}\,\mathrm{Ch}_n(\mathcal{H}, F)(e, e, \dots, e)$$

for n large enough and even. For $[u] \in K_1(\mathcal{A})$,

$$\begin{aligned}\langle [u], [(\mathcal{H}, F)] \rangle &= -\,\mathrm{Ch}_*(\mathcal{H}, F)(u) \\ &:= -\frac{1}{\sqrt{2i}\,2^n\,\Gamma(n/2 + 1)}\,\mathrm{Ch}_n(\mathcal{H}, F)(u^*, u, \dots, u)\end{aligned}$$

for n large enough and odd.

Proof. For the even case the proof, and the result, is just as in [32], and the strategy in the odd case is also the same. However, we present the proof in the odd case in order to clarify the sign convention mentioned above. A similar proof can be given starting from [18], Corollary 3.3.

Using a simple modification of [51], Proposition 4.2, we know that

$$\mathrm{index}(PuP) = \mathrm{trace}((P - Pu^*PuP)^n) - \mathrm{trace}((P - PuPu^*P)^n),$$

where $n > (p+1)/2$ is an integer. First observe that $P - Pu^*PuP = -P[u^*, P]uP$. By replacing P by $(1 + F)/2$ we have

$$P[u^*, P]uP = \tfrac{1}{4}[F, u^*][F, u]\tfrac{1+F}{2}.$$

Since $F[F,a] = -[F,a]F$ for all $a \in \mathcal{A}$, cycling a single $[F,u^*]$ around using the trace property yields

$$\begin{aligned}\operatorname{index}(PuP) &= \operatorname{trace}((P - Pu^*PuP)^n) - \operatorname{trace}((P - PuPu^*P)^n)\\ &= \operatorname{trace}((-\tfrac{1}{4}[F,u^*][F,u]\tfrac{1+F}{2})^n) - \operatorname{trace}((-\tfrac{1}{4}[F,u][F,u^*]\tfrac{1+F}{2})^n)\\ &= (-1)^n \tfrac{1}{4^n} \operatorname{trace}(\tfrac{1+F}{2}([F,u^*][F,u])^n\\ &\qquad - [F,u^*][F,u][F,u^*]\tfrac{1+F}{2}[F,u][F,u^*]\ldots\tfrac{1+F}{2}[F,u]\tfrac{1-F}{2})\\ &= (-1)^n \tfrac{1}{4^n} \operatorname{trace}((\tfrac{1+F}{2} - \tfrac{1-F}{2})([F,u^*][F,u])^n)\\ &= (-1)^n \tfrac{1}{4^n} \operatorname{trace}(F([F,u^*][F,u])^n)\\ &= (-1)^n \tfrac{1}{2^{2n-1}} \operatorname{trace}'(u^*[F,u]\ldots[F,u^*][F,u]),\end{aligned}$$

where in the last line there are $2n-1$ commutators. Comparing the normalisation for the index pairing in [32] and the formula above we find

$$\operatorname{index}(PuP) = -\frac{1}{\sqrt{2i}\,2^n\Gamma(n/2+1)} \operatorname{Ch}_n(\mathcal{H}, F)(u).$$

An independent check can be made on the circle, using the unitary u given by multiplication by $e^{i\theta}$ on $L^2(S^1)$ and the Dirac operator $\frac{1}{i}\frac{d}{d\theta}$. In this case $\operatorname{index}(PuP) = -1$. □

For the reader's benefit, we note that in [22] the signs used are all correct, however in [24] we introduced an additional minus sign (in error) in the formula for the odd case. This disguised the fact that we were not taking a homotopy to the Chern character (as defined above) but rather to minus the Chern character.

The pairing only depends on the K-theory class of e or u and the K-homology class of $(\mathcal{H}, F)$. Whilst the Chern character in this form is very useful for proving basic facts about the index pairing, and relating it to cyclic cohomology, it is not the most computable form for examples.

Imagine trying to compute the pairing of the Hodge–de Rham Fredholm module with K-theory this way. First take the Hilbert space of L^2 differential forms, then construct an invertible version as in Example 17,

$$\mathcal{D}_m = \begin{pmatrix} d + d^* & m \\ m & -(d + d^*) \end{pmatrix},$$

and then take the phase $F = \mathcal{D}_m|\mathcal{D}_m|^{-1}$. Now take commutators with a projection, multiply the commutators together and take a trace. This would appear to be (and is!) much harder to compute with than simply differentiating functions. The issue of finding a computable approach to the index pairing is the most fundamental reason for being interested in spectral triples. They capture the index pairings that arise naturally from the study of unbounded (differential) operators.

5 Spectral triples and computation of the index pairing

5.1 Regularity of spectral triples and algebras. In order to express the pairing of the K-homology class $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ with K-theory directly in terms of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we need more constraints. In particular, to compare our computations with the Chern character computations, we will need to know that $(\mathcal{H}, \mathcal{D}(1 + \mathcal{D}^2)^{-1/2})$ is a finitely summable Fredholm module. On the other hand regularity is about having a noncommutative analogue of differentiability for elements of our algebra. To ensure that a spectral triple represents a K-homology class with a finitely summable representative, we need a summability assumption on the spectral triple, and some regularity as well. The interplay between regularity(=differentiability) and summability(=dimension) is more complicated than in the commutative case.

Definition 5.1. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathrm{Q}C^k$ for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{B}(\mathcal{H})$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathrm{Q}C^\infty$ if it is $\mathrm{Q}C^k$ for all $k \geq 1$.

Remark. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{B}(\mathcal{H})$ which need not be (weakly) dense in $\mathcal{B}(\mathcal{H})$. More precisely, $\operatorname{dom}\delta = \{T \in \mathcal{B}(\mathcal{H}) : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{B}(\mathcal{H})$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1 + \mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{B}(\mathcal{H})$. In fact, writing $|\mathcal{D}|_1 = (1 + \mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\operatorname{dom}\delta^n = \operatorname{dom}\delta_1^n \quad \text{for all } n.$$

Thus the condition defining $\mathrm{Q}C^\infty$ can be replaced by

$$a, [\mathcal{D}, a] \in \bigcap_{n \geq 0} \operatorname{dom}\delta_1^n \quad \text{for all } a \in \mathcal{A}.$$

This is important in situations where we cannot assume that $|\mathcal{D}|$ is invertible.

We saw in Proposition 4.4 that spectral triples define Fredholm modules. In order that a spectral triple defines a finitely summable Fredholm module, and so a Chern character, we need finite summability of the spectral triple, which we take up in the next section. We finish this section with some definitions and results for special algebras that arise in the context of spectral triples.

Definition 5.2. A Fréchet algebra is a locally convex, metrizable and complete topological vector space with jointly continuous multiplication.

We will always suppose that we can define the Fréchet topology of $\mathcal{A}$ using a countable collection of submultiplicative seminorms which includes the C*-norm of $\bar{\mathcal{A}} = A$, and note that the multiplication is jointly continuous. By replacing any seminorm q by $\frac{1}{2}(q(a) + q(a^*))$, we may suppose that $q(a) = q(a^*)$ for all $a \in \mathcal{A}$.

Definition 5.3. A subalgebra $\mathcal{A}$ of a C*-algebra A is a *pre-C*-algebra* or *stable under the holomorphic functional calculus* if, whenever $a \in \mathcal{A}$ is invertible in A, it is invertible in $\mathcal{A}$. Equivalently, $\mathcal{A}$ is a pre-C*-algebra if, whenever $f : \mathbb{C} \to \mathbb{C}$ is a function holomorphic in a neighbourhood of the spectrum of $a \in \mathcal{A}$, then the element $f(a) \in A$ defined by the continuous functional calculus is in fact in $\mathcal{A}$, i.e., $f(a) \in \mathcal{A}$.

Definition 5.4. A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C*-algebra A which is stable under the holomorphic functional calculus.

Thus, saying that $\mathcal{A}$ is smooth means that $\mathcal{A}$ is Fréchet and a pre-C*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C*-topology of A (since Fréchet means locally convex, metrizable and complete.)

Remark. It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [51], [85]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. This definition ensures that a smooth algebra is a ‘good’ algebra, [51], so these algebras have a sensible spectral theory which agrees with that defined using the C*-closure, and that the group of invertibles is open.

Lemma 5.5 ([51], [79]). *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a* QC$^\infty$ *spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a* QC$^\infty$ *spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{ni}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0,\ i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

Thus, whenever we have a QC$^\infty$ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may suppose without loss of generality that the algebra $\mathcal{A}$ is a Fréchet pre-C*-algebra. Thus $\mathcal{A}$ suffices to capture all the K-theory of A. This is necessary if we are to use spectral triples to compute the index pairing. A QC$^\infty$ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for which $\mathcal{A}$ is complete has not only a holomorphic functional calculus for $\mathcal{A}$ but also a C^∞ *functional calculus* for self-adjoint elements: we quote [79], Proposition 22.

Proposition 5.6 (C^∞ Functional Calculus). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a* QC$^\infty$ *spectral triple, and suppose $\mathcal{A}$ is complete in the δ-topology. Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function in a neighbourhood of the spectrum of $a = a^* \in \mathcal{A}$. If we define $f(a) \in A$ using the continuous functional calculus, then in fact $f(a)$ lies in $\mathcal{A}$.* □

Remark. For each $a = a^* \in \mathcal{A}$, the C^∞-functional calculus defines a continuous homomorphism $\Psi\colon C^\infty(U) \to \mathcal{A}$ where $U \subset \mathbb{R}$ is any open set containing the spectrum of a, and the topology on $C^\infty(U)$ is that of uniform convergence of all derivatives on compact subsets.

5.2 Summability for spectral triples

5.2.1 Finite summability. Just as for Fredholm modules, we require a notion of summability for spectral triples as this is needed to write down explicit formulae for index pairings and representatives of the Chern character.

Definition 5.7. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called finitely summable if there is some $s_0 > 0$ such that

$$\operatorname{trace}((1+\mathcal{D}^2)^{-s_0/2}) < \infty.$$

This is then true for all $s > s_0$ and we call

$$p = \inf\{s \in \mathbb{R}_+ : \operatorname{trace}((1+\mathcal{D}^2)^{-s/2}) < \infty\}$$

the spectral dimension.

Remark. What finitely summable means for a spectral triple with $\mathcal{A}$ non-unital and $(1+\mathcal{D}^2)^{-1/2}$ not a compact operator, but of course with $a(1+\mathcal{D}^2)^{-1/2}$ compact for all $a \in \mathcal{A}$, is still an open question; see [47], [79], [80].

Not all algebras have finitely summable spectral triples, even when they have finitely summable Fredholm modules (more on this later). We quote the following necessary condition.

Theorem 5.8 (Connes, [33]). *Let A be a unital C^*-algebra and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a finitely summable QC^1 spectral triple, with $\mathcal{A} \subset A$ dense. Then there exists a positive trace τ on A with $\tau(1) = 1$.*

So algebras with no normalised trace, such as the Cuntz algebras, do not have finitely summable spectral triples associated to them.

Proposition 5.9. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a finitely summable QC^1 spectral triple with spectral dimension $p \geq 0$, then $(\mathcal{H}, F_{\mathcal{D}})$ is a $([p]+1)$-summable Fredholm module for $\mathcal{A}$, where $[p]$ is the largest integer less than or equal to p.*

Proof. Let $a \in \mathcal{A}$ and recall

$$[F_{\mathcal{D}}, a] = [\mathcal{D}, a](1+\mathcal{D}^2)^{\frac{-1}{2}} - F_{\mathcal{D}}[(1+\mathcal{D}^2)^{\frac{1}{2}}, a](1+\mathcal{D}^2)^{\frac{-1}{2}} =: T(1+\mathcal{D}^2)^{\frac{-1}{2}}.$$

Now observe that T is bounded, and we want to show that

$$T(1+\mathcal{D}^2)^{\frac{-1}{2}} T(1+\mathcal{D}^2)^{\frac{-1}{2}} \dots T(1+\mathcal{D}^2)^{\frac{-1}{2}} \in \mathcal{L}^1(\mathcal{H}),$$

where we have a product of $[p]+1$ terms. For each $\varepsilon > 0$ we have $T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p+\varepsilon}(\mathcal{H})$. As $[p] \le p < [p]+1$, we can choose $p+\varepsilon$ between p and $[p]+1$, and so $T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{[p]+1}(\mathcal{H})$. The product is in $\mathcal{L}^1(\mathcal{H})$ and we are done. □

Remark. As in Proposition 4.4, using [17] we can replace $\mathrm{Q}C^1$ by $\mathrm{Q}C^0$.

The finitely summable Fredholm module above is not normalised. To obtain a normalised finitely summable Fredholm module, we follow the same recipe that we applied to the Hodge–de Rham example.

Lemma 5.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. For any $m > 0$ we define the double of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be the spectral triple $(\mathcal{A}, \mathcal{H}_2, \mathcal{D}_m)$ with*

$$\mathcal{H}_2 = \mathcal{H} \oplus \mathcal{H}, \quad \mathcal{D}_m = \begin{pmatrix} \mathcal{D} & m \\ m & -\mathcal{D} \end{pmatrix}, \quad a \to \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is graded by γ, the double is graded by $\gamma \oplus -\gamma$. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathrm{Q}C^k$, $k = 0, 1, \dots, \infty$, so is the double. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension p, the double is finitely summable with spectral dimension p. Moreover, the K-homology classes of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and its double coincide for any $m > 0$. This class can be represented by the normalised Fredholm module $(\mathcal{H}_2, \mathcal{D}_m|\mathcal{D}_m|^{-1})$.

Remark. The explicit identification of the K-homology classes and the normalised representative can be found in [21].

5.2.2 (n, ∞)-summability and the Dixmier trace.

We recall the classical result of Weyl.

Theorem 5.11 (Weyl's theorem). *Let P be an order d elliptic differential operator on a compact oriented manifold M of dimension n. Let $\{\lambda_k\}$ denote the eigenvalues of P ordered so that $|\lambda_1| \le |\lambda_2| \le \cdots$ and repeated according to multiplicity. Then*

$$|\lambda_k| \sim C k^{\frac{d}{n}} + O(k^{\frac{d}{n}-1}).$$

The constant C can also be computed, but we will leave that for a little while. First we will introduce some analytic machinery. If $T \in \mathcal{K}(\mathcal{H})$, let $\mu_n(T)$ denote the n-th singular number of T; that is, $\mu_n(T)$ is the n-th eigenvalue of $\sqrt{T^*T}$ when they are listed in nonincreasing order and repeated according to multiplicity. Let

$$\sigma_N(T) = \sum_{k=1}^{N} \mu_k(T)$$

be the N-th partial sum of the singular values. For $p > 1$ let

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(N^{1-\frac{1}{p}})\}$$

and for $p = 1$ let

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(\log N)\}.$$

We will be interested in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$; however the following is useful: If $T_1, \ldots, T_m$ are in $\mathcal{L}^{(p_1,\infty)}, \ldots, \mathcal{L}^{(p_m,\infty)}$, respectively, and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, then $T_1 T_2 \ldots T_m \in \mathcal{L}^{(1,\infty)}$. While the Schatten classes play a similar role to the L^p spaces of classical analysis, the $\mathcal{L}^{(p,\infty)}$ spaces play a role similar to weak L^p spaces.

What we would like to do is construct a functional on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ by defining, for $T \geq 0$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

However, this formula need not define a linear functional, and it may not converge. The trick is to consider the sequence

$$\Big(\frac{\sigma_2(T)}{\log 2}, \frac{\sigma_3(T)}{\log 3}, \frac{\sigma_4(T)}{\log 4}, \ldots\Big)$$

and observe that this sequence is bounded. If it always converged, the limit would provide a linear functional which is a trace. As it does not always converge, we must consider certain generalised limits, whose explicit definition can be found in [25]. We will denote by $\lim_\omega$ any such generalised limit, and observe that there are uncountably many such, as is explained in [25], [34]. The resulting functionals on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ are called Dixmier traces.

Proposition 5.12. *For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ and suitable $\omega \in \ell^\infty(\mathbb{N})^*$ we define the associated Dixmier trace*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

Then:

(1) $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$*, so we can extend it by linearity to all of* $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.

(2) *If* $T \geq 0$*, then* $\mathrm{Tr}_\omega(T) \geq 0$.

(3) *If* $S \in \mathcal{B}(\mathcal{H})$ *and* $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$*, then* $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.

Moreover, for any trace class operator T we have $\mathrm{Tr}_\omega(T) = 0$.

While the above result holds for a range of functionals ω, in practise the value of a Dixmier trace on operators that arise in examples is independent of the choice of ω: we call such operators measurable.

Here is the key criterion for measurability. First, for $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, $T \geq 0$, define

$$\zeta_T(s) = \text{trace}(T^s) = \sum_{k=1}^{\infty} \mu_k(T)^s$$

for $\text{Re}(s) > 1$. Then we have:

Proposition 5.13. *With $T \geq 0$ as above the following are equivalent:*

(1) $(s-1)\zeta_T(s) \to L$ *as* $s \searrow 1$;

(2) $\frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T) \to L$ *as* $N \to \infty$.

In this case, the residue at $s = 1$ of $\zeta_T(s)$ is precisely $\text{Tr}_\omega(T)$ and so the value of Tr_ω on T is independent of ω. In fact, non-measurable operators are not natural; see [51] for an example.

Proposition 5.14 (Connes' trace theorem). *Let M be an n-dimensional compact manifold and let T be a classical pseudodifferential operator of order $-n$ (think of $T = (1 + \mathcal{D}^2)^{-n/2}$ where $\mathcal{D}$ is of order 1) acting on sections of a complex vector bundle $E \to M$. Then:*

(1) *The corresponding operator T on $L^2(M, E)$ belongs to the ideal $\mathcal{L}^{(1,\infty)}$.*

(2) *The Dixmier trace $\text{Tr}_\omega(T)$ is independent of ω and is equal to the Wodzicki residue:*

$$\text{WRes}(T) = \frac{1}{n(2\pi)^n} \int_{S^*M} \text{trace}_E(\sigma_T(x, \xi))\, d\,\text{vol}.$$

Here S^*M is the cosphere bundle, $\{\xi \in T^*M : \|\xi\|^2 = g^{\mu\nu}\xi_\mu\xi_\nu = 1\}$.

The surprising fact about the Wodzicki residue is that it extends to a trace (the unique such trace) on the whole algebra of pseudodifferential operators of any order. This extension is simply to take the $-n$-th part of the symbol and integrate it over the co-sphere bundle. Thus the residue of the zeta function can be computed geometrically.

In the following we restrict attention to operators 'of Dirac type', by which we mean that the principal symbol of $\mathcal{D}$ is given by Clifford multiplication. This means that the symbol of $\mathcal{D}^2$ is given by $\sigma_{\mathcal{D}^2}(x, \xi) = \|\xi\|^2$.

Corollary 5.15. *Let $f \in C^\infty(M)$ and $\mathcal{D}$ be a first order self-adjoint elliptic operator 'of Dirac type' on the vector bundle E. Then the operator $f(1 + \mathcal{D}^2)^{-n/2}$ acting on $L^2(E)$ is measurable and*

$$\text{Tr}_\omega(f(1 + \mathcal{D}^2)^{-n/2}) = \frac{\text{rank}(E)\,\text{vol}(S^{n-1})}{n(2\pi)^n} \int_M f\, d\,\text{vol}.$$

Hence the representation of functions as multiplication operators, along with the spectrum of $\mathcal{D}$, is enough to recover the integral on a manifold using the Dixmier trace on smooth functions. For a clear discussion of what happens for measurable functions, see [28], [67]. This has stimulated interest in other spectral triples satisfying the following summability hypothesis.

Definition 5.16. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is (n, ∞)-summable if

$$(1 + \mathcal{D}^2)^{-n/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

This definition is definitely in the context of unital algebras $\mathcal{A}$. For an approach to this definition when $\mathcal{A}$ is non-unital see [47], [79], [80]. Observe that if a spectral triple is (n, ∞)-summable, then the associated Fredholm module is $(n + 1)$-summable. Also, the spectral dimension of such a triple is n.

Example 18. Examining the eigenvalues of the 'Dirac' operator for the noncommutative torus in Section 3.5 (for simplicity set $\tau = i$), we see that the eigenvalues obey Weyl's theorem. This is not surprising since $\mathcal{D}$ and $\mathcal{H}$ are actually the same as in the commutative case. Hence the spectral triple for the noncommutative torus is $(2, \infty)$-summable with spectral dimension $p = 2$.

Exercise. Compute the Dixmier trace of $(1 + \mathcal{D}^2)^{-1}$ for the noncommutative torus. *Hint*: See [51].

Example 19. The Cantor set spectral triple introduced in Example 7 is also illuminating. If the gap between e_- and e_+ appears at the n-th stage of our construction (counting the interval $[0, 1]$ as the 0-th stage), then $e_+ - e_- = 3^{-n}$. How many gaps are there with this length? The answer is 2^{n-1}, except for $n = 0$. Including the extra 2 for the 2×2 matrix structure of $|\mathcal{D}|$, we find that the trace of $|\mathcal{D}|^{-s}$ for $s \gg 1$ is

$$\zeta(s) = 2 + \sum_{n=1}^{\infty} 2^n 3^{-ns} = 1 + \frac{1}{1 - 2/3^s}.$$

This is finite for $s > \frac{\log 2}{\log 3}$ and this formula provides a meromorphic continuation of $\zeta(s)$ whose only singularities are simple poles at $s = (\log 2 + 2k\pi i)/\log 3$. The number $\log 2/\log 3$ is the Hausdorff dimension of the Cantor set.

Exercise. Find the residue at $s = \log 2/\log 3$.

The relationship between the Dixmier trace and the zeta function, as well as the heat kernel asymptotics, is well described in [25], and the definitive results are in [26] and [67].

5.2.3 θ-summability

Definition 5.17. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable if for all $t > 0$ we have

$$\operatorname{trace}(e^{-t\mathcal{D}^2}) < \infty.$$

Example 20. There are many interesting spectral triples that are not finitely summable. This has the consequence that our main tool (which we describe later), the local index formula, is not available. The examples arising from supersymmetric quantum field theory are not finitely summable, but rather than take a detour into physics, we will look at examples coming from group C*-algebras. All of this material, plus the construction of the metric on the state space first appeared in the beautiful paper [33].

Let Γ be a finitely generated group, and let $\mathbb{C}\Gamma$ denote the group ring of Γ. The group ring $\mathbb{C}\Gamma$ acts by bounded operators on the Hilbert space $l^2(\Gamma)$. The action is defined on the dense linear subspace $\mathbb{C}\Gamma \subset l^2(\Gamma)$ by left multiplication. One shows that each $a \in \mathbb{C}\Gamma$ extends to a bounded operator on $l^2(\Gamma)$ and that we obtain a $*$-homomorphism $\mathbb{C}\Gamma \to \mathcal{B}(l^2(\Gamma))$. This is called the left regular representation.

Then the C*-algebra $C^*_{\mathrm{red}}(\Gamma)$, called the reduced C*-algebra of Γ, is the norm closure of the image of $\mathbb{C}\Gamma$ in $\mathcal{B}(l^2(\Gamma))$. For $\psi \in l^2(\Gamma)$ and $g \in \Gamma$, the left regular representation is given by

$$(\lambda(g)\psi)(k) = \psi(g^{-1}k).$$

Define a length function on Γ to be a function $L\colon \Gamma \to \mathbb{R}^+$ such that

1) $L(gh) \leq L(g) + L(h)$ for all $g, h \in \Gamma$,
2) $L(g^{-1}) = L(g)$ for all $g \in \Gamma$,
3) $L(1) = 0$.

The prototypical example is the word length function. Let $G \subset \Gamma$ be a generating set. Then for all $g \in \Gamma$, $g = g_1 \dots g_n$ for some n, where $g_i \in G$ for all $i = 1, \dots, n$. This expression is not unique, and we define

$$L(g) = \min\{n : g = g_1 \dots g_n,\ g_i \in G,\ i = 1, \dots, n\}.$$

Using length functions we can construct spectral triples.

Lemma 5.18. *Let Γ be a finitely generated discrete group and L a length function on Γ. Let $\mathcal{D}$ be the operator of multiplication by L on $\mathcal{H} = l^2(\Gamma)$. If L is unbounded on Γ, then*

(1) *$(\mathbb{C}(\Gamma), \mathcal{H}, \mathcal{D})$ is a spectral triple,*
(2) *$\|[\mathcal{D}, \lambda(g)]\| = L(g)$ for all $g \in \Gamma$.*

Proof. To show that for a dense subalgebra $\mathcal{A} \subset C^*(\Gamma)$ the commutators $[\mathcal{D}, a]$ are bounded, it suffices to show that for all $g \in \Gamma$, the commutator $[\mathcal{D}, \lambda(g)]$ is bounded (the group ring $\mathbb{C}\Gamma$ is dense). We compute

$$\begin{aligned}(\mathcal{D}\lambda(g)\psi)(k) - (\lambda(g)\mathcal{D}\psi)(k) &= \mathcal{D}\psi(g^{-1}k) - \lambda(g)L(k)\psi(k) \\ &= L(g^{-1}k)\psi(g^{-1}k) - L(k)\psi(g^{-1}k).\end{aligned}$$

However

$$|L(g^{-1}k) - L(k)| \leq |L(g^{-1}) + L(k) - L(k)| = L(g),$$

so this is bounded.

Now for any real number $x \in \mathbb{R}$, let $K_x \subset \Gamma$ be those group elements with $L(g) = x$. Let ψ_x be the function in $l^2(\Gamma)$ with $\psi_x \equiv 1$ on K_x and zero elsewhere. Then

$$\mathcal{D}\psi_x = x\psi_x.$$

As Γ is discrete, L takes on only a discrete set of values. Thus there are a countable number of ψ_xs and corresponding eigenvalues x. With the assumption that $L(g) \to \infty$ as $g \to \infty$, we see that $\mathcal{D}$ is unbounded, has countably many eigenvalues of finite multiplicity, and this is enough to conclude that $\mathcal{D}$ has compact resolvent. This shows that we have a spectral triple. We do not know if it is even or odd.

Lastly, let ψ_1 be the function which is 1 on $1 \in \Gamma$ and zero elsewhere. Then

$$\begin{aligned}([\mathcal{D}, \lambda(g)]\psi_1)(k) &= (\mathcal{D}\lambda(g)\psi_1)(k) - (\lambda(g)\mathcal{D}\psi_1)(k) \\ &= (\mathcal{D}\psi_1)(g^{-1}k) - (\lambda(g)L(k)\psi_1)(k) \\ &= (L(g^{-1}k) - L(k))\psi_1(g^{-1}k) \\ &= \begin{cases} 0, & k \neq g^{-1}, \\ -L(k), & k = g^{-1}, \end{cases} \\ &= -L(g)\delta_{k,g^{-1}}.\end{aligned}$$

So, as we showed that $\|[\mathcal{D}, \lambda(g)]\| \leq L(g)$, the above calculation shows that equality always holds, proving the second assertion of the lemma. □

So we have a spectral triple, a priori odd (ungraded). We are interested in seeing whether it is finitely summable.

Theorem 5.19 (Connes, [33]). *Let Γ be a discrete group containing the free group on two generators. Let $\mathcal{H}$ be any representation of $C^*(\Gamma)$, absolutely continuous with respect to the canonical trace on $C^*(\Gamma)$. Then there does not exist a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that $(\mathcal{H}, \mathcal{D})$ is a finitely summable spectral triple for $C^*_{\mathrm{red}}(\Gamma)$.*

Remark. There may be finitely summable Fredholm modules for such a group algebra. In particular, one is known for the free group on two generators. The culprit here is the lack of hyperfiniteness of the group von Neumann algebra.

Theorem 5.20 (Connes, [33]). *If Γ is an infinite discrete group with property T, then there exists no finitely summable spectral triple for $C^*_{\mathrm{red}}(\Gamma)$.*

Again, there are interesting finitely summable Fredholm modules for such groups but the fundamental problem is that some group C*-algebras are 'infinite dimensional noncommutative spaces'.

Theorem 5.21 (Connes, [33]). *Let Γ be a finitely generated discrete group, and L the word length function, relative to some generating subset. Let $\mathcal{H} = l^2(\Gamma)$, with $C^*_{\mathrm{red}}(\Gamma)$ acting by multiplication and let $\mathcal{D}$ be multiplication by the word length function L. Then $(\mathcal{H}, \mathcal{D})$ is a θ-summable spectral triple for $C^*_{\mathrm{red}}(\Gamma)$.*

5.3 Analytic formulae for the index. There are analytic formulae for the pairing between a spectral triple and K-theory with no reference to cyclic cohomology. Nevertheless, it is via these formulae that the link to cyclic cohomology is made in [22], [23]. These formulae are truly purely analytic, and apart from a few special cases do not provide in themselves a computational tool. Rather, they allow us to derive a variety of different formulae from which we can connect to, say, topological formulae for the index in the case of manifolds.

Theorem 5.22 (McKean–Singer formula). *Let $\mathcal{D}$ be an unbounded self-adjoint operator with compact resolvent. Let γ be a self-adjoint unitary which anticommutes with $\mathcal{D}$. Finally, let f be a continuous even function on $\mathbb{R}$ with $f(0) \neq 0$ and $f(\mathcal{D})$ trace-class. Let $\mathcal{D}^+ = P^\perp \mathcal{D} P$ where $P = (1+\gamma)/2$ and $P^\perp = 1 - P$. Then $D^+ \colon P\mathcal{H} \to P^\perp\mathcal{H}$ is Fredholm and*

$$\operatorname{index}(D^+) = \frac{1}{f(0)} \operatorname{trace}(\gamma f(D)). \tag{5.1}$$

This version (actually a stronger version valid in semfinite von Neumann algebras) can be found in [23]. The traditional function used in this context is $f(x) = e^{-tx^2}$, $t > 0$, so the formula becomes

$$\operatorname{index}(\mathcal{D}^+) = \operatorname{trace}(\gamma e^{-t\mathcal{D}^2}).$$

The operator $e^{-t\mathcal{D}^2}$ is called the *heat operator*, being the solution of the 'heat equation' $\partial_t A(t) + \mathcal{D}^2 A(t) = 0$. However for a finitely summable spectral triple, functions such as $f(x) = (1+x^2)^{-s/2}$ for s large provide a natural alternative. As we remarked previously, in the case of odd spectral triples the pairing between K-theory and K-homology calculates spectral flow.

Theorem 5.23 (Spectral flow formula,[17], [18]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable spectral triple with spectral dimension $p \geq 1$. Let $u \in \mathcal{A}$ be unitary and let P be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$. Then for any $s > p$ the spectral flow along the line segment joining $\mathcal{D}$ to $u\mathcal{D}u^*$ is given by*

$$\operatorname{index}(PuP) = \frac{1}{C_{s/2}} \int_0^1 \operatorname{trace}(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2})\, dt, \tag{5.2}$$

with $C_{s/2} = \int_{-\infty}^{\infty} (1+x^2)^{-s/2} dx$.

Both of the analytic formulae are scale invariant. By this we mean that if we replace $\mathcal{D}$ by $\varepsilon\mathcal{D}$, for $\varepsilon > 0$, in the right-hand side of (5.2) or (5.1), then the left-hand side is unchanged since in both cases the index is invariant with respect to change of scale. Rewriting the 'constant' $C_{s/2}$ as

$$C_{s/2} = \frac{\Gamma(s - 1/2)\Gamma(1/2)}{\Gamma(s)}$$

we see that in fact the integral formula in (5.2) can be given a meromorphic continuation (as a function of s) by setting

$$\text{index}(PuP)C_{s/2} = \int_0^1 \text{trace}(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2})\, dt.$$

Here we have written the right-hand side in bold face to indicate that we are thinking of the meromorphically continued function. Since the residue of $C_{s/2}$ at $s = 1/2$ is 1, we also have

$$\text{index}(PuP) = \text{res}_{s=1/2} \int_0^1 \text{trace}(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2})\, dt.$$

This observation is the starting point for the proof of the local index formula in [22]. A suitable choice of functions allows a similar analysis in the even case; we refer to [23].

6 The Chern character of spectral triples

6.1 Cyclic homology and cohomology. A central feature of [34] is the expression of the K-theory pairing in terms of cyclic theory in order to obtain index theorems. In this section we will summarise the relevant notions of cyclic theory.

The aim is to associate to a suitable representative of a K-theory class, respectively a K-homology class, a class in periodic cyclic homology, respectively a class in periodic cyclic cohomology, called a Chern character in both cases. The principal result is then

$$\langle [x], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\frac{1}{\sqrt{2\pi i}} \langle [\text{Ch}_*(x)], [\text{Ch}^*(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle,$$

where $[x] \in K_*(\mathcal{A})$ is a K-theory class with representative x and $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the K-homology class of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

On the right-hand side, $\text{Ch}_*(x)$ is the Chern character of x, and $[\text{Ch}_*(x)]$ its cyclic homology class. Similarly $[\text{Ch}^*(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the cyclic cohomology class of a representative of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

We will use the normalised (b, B)-bicomplex (see [34], [66]). The reason for this is that one can easily realise the Chern character of a finitely summable Fredholm module, a cyclic cocycle, in the (b, B) picture, but going the other way requires substantial work, [24].

We introduce the following linear spaces. Let $C_m = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ where $\bar{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathbb{C}I$ with I being the identity element of $\mathcal{A}$ and (assuming with no loss of generality that $\mathcal{A}$ is complete in the δ-topology) we employ the projective tensor product. Let $C^m = \text{Hom}(C_m, \mathbb{C})$ be the linear space of continuous multilinear functionals on C_m. We may define the (b, B)-bicomplex using these spaces (as opposed to $C_m = \mathcal{A}^{\otimes m+1}$ etc) and the resulting cohomology will be the same. This follows because the bicomplex defined using $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ is quasi-isomorphic to that defined using $\mathcal{A} \otimes \mathcal{A}^{\otimes m}$.

A normalised (b, B)-*cochain*, ϕ, is a finite collection of continuous multilinear functionals on $\mathcal{A}$,

$$\phi = (\phi_m)_{m=1,2,\dots,M} \quad \text{with } \phi_m \in C^m.$$

It is a (normalised) (b, B)-*cocycle* if, for all m, $b\phi_m + B\phi_{m+2} = 0$, where $b\colon C^m \to C^{m+1}$, $B\colon C^m \to C^{m-1}$ are the coboundary operators given by

$$(B\phi_m)(a_0, a_1, \dots, a_{m-1}) = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \phi_m(1, a_j, a_{j+1}, \dots, a_{m-1}, a_0, \dots, a_{j-1}),$$

$$(b\phi_{m-2})(a_0, a_1, \dots, a_{m-1}) = \sum_{j=0}^{m-2} (-1)^j \phi_{m-2}(a_0, a_1, \dots, a_j a_{j+1}, \dots, a_{m-1}) + (-1)^{m-1} \phi_{m-2}(a_{m-1}a_0, a_1, \dots, a_{m-2}),$$

We write $(b + B)\phi = 0$ for brevity. Thought of as functionals on $\mathcal{A}^{\otimes m+1}$ a normalised cocycle will satisfy $\phi(a_0, a_1, \dots, a_n) = 0$ whenever any $a_j = 1$ for $j \geq 1$. An *odd* (*even*) cochain has $\phi_m = 0$ for m even (odd).

Similarly, a (b^T, B^T)-*chain*, c is a (possibly infinite) collection $c = (c_m)_{m=1,2,\dots}$ with $c_m \in C_m$. The (b, B)-chain (c_m) is a (b^T, B^T)-*cycle* if $b^T c_{m+2} + B^T c_m = 0$ for all m. More briefly, we write $(b^T + B^T)c = 0$. Here b^T, B^T are the boundary operators of cyclic homology, and are the transpose of the coboundary operators b, B in the following sense.

The pairing between a (b, B)-cochain $\phi = (\phi_m)_{m=1}^M$ and a (b^T, B^T)-chain $c = (c_m)$ is given by

$$\langle \phi, c \rangle = \sum_{m=1}^{M} \phi_m(c_m) \quad (M \in \mathbb{N} \text{ or } M = \infty).$$

This pairing satisfies

$$\langle (b + B)\phi, c \rangle = \langle \phi, (b^T + B^T)c \rangle.$$

We use this fact in the following way. We call $c = (c_m)_{m\,\text{odd}}$ an odd normalised (b^T, B^T)-*boundary* if there is some even chain $e = (e_m)_{m\,\text{even}}$ with $c_m = b^T e_{m+1} + B^T e_{m-1}$ for all m. If we pair a normalised (b, B)-cocycle ϕ with a normalised (b^T, B^T)-boundary c we find

$$\langle \phi, c \rangle = \langle \phi, (b^T + B^T)e \rangle = \langle (b + B)\phi, e \rangle = 0.$$

There is an analogous definition in the case of even chains $c = (c_m)_{m\,\text{even}}$. All of the cocycles we consider in these notes are in fact defined as functionals on $\bigoplus_m \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$. Henceforth we will drop the superscript on b^T, B^T and just write b, B for both boundary and coboundary operators as the meaning will be clear from the context.

Recall that the Chern character $\mathrm{Ch}_*(u)$ of a unitary $u \in \mathcal{A}$ is the following (infinite) collection of odd chains $\mathrm{Ch}_{2j+1}(u)$ satisfying $b\,\mathrm{Ch}_{2j+3}(u) + B\,\mathrm{Ch}_{2j+1}(u) = 0$:

$$\mathrm{Ch}_{2j+1}(u) = (-1)^j j!\, u^* \otimes u \otimes u^* \otimes \cdots \otimes u \quad (2j + 2 \text{ entries}).$$

Exercise. Check that $\mathrm{Ch}_*(u)$ is a (b, B)-cycle, and that $\mathrm{Ch}_*(u) + \mathrm{Ch}_*(u^*)$ is a coboundary.

Similarly, the (b, B) Chern character of a projection p in an algebra $\mathcal{A}$ is an even (b, B) cycle with $2m$-th term, $m \geq 1$, given by

$$\mathrm{Ch}_{2m}(p) = (-1)^m \tfrac{(2m)!}{2(m!)}(2p-1) \otimes p^{\otimes 2m}.$$

For $m = 0$ the definition is $\mathrm{Ch}_0(p) = p$.

Exercise. Check that $\mathrm{Ch}_*(p)$ is a (b, B)-cycle.

Since the (b, B) Chern character of a projection or unitary has infinitely many terms which grow rapidly, we need some constraint on the cochains we pair them with. If we allow only finitely supported cochains, then we obtain the usual cyclic cohomology groups $\mathrm{HC}^*(\mathcal{A})$. The Chern character of a finitely summable spectral triple is finitely supported.

If we allow infinitely supported cochains which satisfy some decay condition α, then we get something we shall call $\mathrm{HC}^*_\alpha(\mathcal{A})$. The most commonly used condition is to look at entire cochains, and the reason for this is that the JLO cocycle is entire; see [34]. Very often one finds that for any reasonable decay condition α we have $\mathrm{HC}^*_\alpha(\mathcal{A}) \cong \mathrm{HC}^*(\mathcal{A})$, but general statements are hard to find.

A final warning: cyclic (co)homology of a C^*-algebra is trivial. It is necessary to work with a smooth subalgebra, or employ a cyclic theory developed for C^*-algebras called local cyclic (co)homology, due to Puschnigg, [76]. Alternatively, one could use Kasparov's KK-theory with smooth algebras. This approach is developed by Cuntz, [43]. In general, however, the tension between continuous and smooth theories appears naturally.

6.2 The local index formula for finitely summable smooth spectral triples. A natural question is whether there is an analogue of the Atiyah–Singer index theorem in noncommutative geometry. This question was answered, in a sense, by Connes and Moscovici in the paper [40]. What the local index formula of Connes and Moscovici provides is a schematic: a general formula which must still be massaged further to get interesting information for particular examples. One example of this process is the proof of the Atiyah–Singer index theorem by Ponge, [75], starting from the Connes–Moscovici residue cocycle. Similarly the Atiyah–Singer index formula has been derived from the JLO cocycle by Block and Fox, [11].

Generalisations and new proofs of Connes and Moscovici's local index formula have been given in [53], [22], [23]. The main point to appreciate is that given a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ there are various representatives of the Chern character. We have seen one already in the form of Connes' definition of the Chern character of the Fredholm module associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In this section we present several more.

The (finite) summability conditions give a half-plane where the function

$$z \mapsto \mathrm{trace}((1+\mathcal{D}^2)^{-z}) \tag{6.1}$$

is well defined and holomorphic. In [35], [40], a stronger condition was imposed in order to prove the local index formula. This condition not only specifies a half-plane where the function in (6.1) is holomorphic, but also that this function analytically continues to $\mathbb{C}$ minus some discrete set. We clarify this in the following definitions.

Definition 6.1. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathrm{Q}C^\infty$ spectral triple. The algebra $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H})$ is the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([\mathcal{D}, a])$ for $a \in \mathcal{A}$ and $n \geq 0$. A $\mathrm{Q}C^\infty$ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has *discrete dimension spectrum* $\mathrm{Sd} \subseteq \mathbb{C}$ if Sd is a discrete set and for all $b \in \mathcal{B}(\mathcal{A})$ the function $\mathrm{trace}(b(1+\mathcal{D}^2)^{-z})$ is defined and holomorphic for $\mathrm{Re}(z)$ large, and analytically continues to $\mathbb{C} \setminus \mathrm{Sd}$. We say the dimension spectrum is *simple* if this zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$, *finite* if there is a $k \in \mathbb{N}$ such that the function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$, and *infinite* if it is not finite.

Connes and Moscovici impose the discrete dimension spectrum assumption to prove their original version of the local index formula. The dimension spectrum idea is quite attractive in a number of respects. The dimension spectrum of a direct sum of spectral triples is the union of the dimension spectra of the summands. The dimension spectrum of a product consists of sums of elements in the dimension spectra of the 'prodands'.

New proofs of the local index formula were presented by Nigel Higson, and by Carey, Phillips, Rennie, Sukochev. These were much simpler, more widely applicable and in the case of [21], [22], required much less restriction on the zeta functions, and in particular did not require the discrete dimension spectrum hypothesis. We will introduce some notation and definitions and then state the local index formula using [21], [22].

Introduce multi-indices $(k_1, \ldots, k_m)$, $k_i = 0, 1, 2, \ldots$, whose length m will always be clear from the context and let $|k| = k_1 + \cdots + k_m$. Define

$$\alpha(k) = \frac{1}{k_1!k_2!\ldots k_m!\,(k_1+1)(k_1+k_2+2)\ldots(|k|+m)}$$

and the numbers $\widetilde{\sigma}_{n,j}$ and $\sigma_{n,j}$ are defined by the equalities

$$\prod_{j=0}^{n-1}(z+j+1/2) = \sum_{j=0}^{n} z^j \widetilde{\sigma}_{n,j} \quad \text{and} \quad \prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} \sigma_{n,j}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a $\mathrm{Q}C^\infty$ spectral triple and $T \in \mathcal{N}$ then $T^{(n)}$ is the n^{th} iterated commutator with $\mathcal{D}^2$, that is, $[\mathcal{D}^2, [\mathcal{D}^2, [\ldots, [\mathcal{D}^2, T]\ldots]]]$.

Definition 6.2. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a $\mathrm{Q}C^\infty$ finitely summable spectral triple, we call

$$q = \inf\{k \in \mathbb{R} : \mathrm{trace}((1+\mathcal{D}^2)^{-k/2}) < \infty\}$$

the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension if for all b of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \dots [\mathcal{D}, a_m]^{(k_m)}(1 + \mathcal{D}^2)^{-m/2-|k|}, \quad a_j \in \mathcal{A},$$

the zeta functions

$$\zeta_b(z - (1-q)/2) = \mathrm{trace}(b(1 + \mathcal{D}^2)^{-z+(1-q)/2})$$

have analytic continuations to a deleted neighbourhood of $z = (1-q)/2$.

Remark. Observe that we allow the possibility that the analytic continuations of these zeta functions may have an essential singularity at $z = (1-q)/2$. All that is necessary for us is that the residues at this point exist. Note that discrete dimension spectrum implies isolated spectral dimension.

Now let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with isolated spectral dimension. For operators $b \in \mathcal{B}(\mathcal{H})$ of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \dots [\mathcal{D}, a_m]^{(k_m)}(1 + \mathcal{D}^2)^{-m/2-|k|}$$

we can define the functionals

$$\tau_j(b) := \mathrm{res}_{z=(1-q)/2}(z - (1-q)/2)^j \zeta_b(z - (1-q)/2).$$

The hypothesis of isolated spectral dimension is clearly necessary here in order to define the residues. Let P be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$.

In [22] we proved the following result:

Theorem 6.3 (Odd local index formula). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $q \geq 1$. Let $N = [q/2] + 1$ where $[\,\cdot\,]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

$$\text{(1)} \qquad \mathrm{index}(PuP) = \tfrac{1}{\sqrt{2\pi i}} \mathrm{res}_{r=(1-q)/2} \Big(\sum_{m=1,\,\mathrm{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}_m(u)) \Big),$$

where for $a_0, \dots, a_m \in \mathcal{A}$, $l = \{a + iv : v \in \mathbb{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ and $r > 0$ we define $\phi_m^r(a_0, a_1, \dots, a_m)$ to be

$$\tfrac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \mathrm{trace}\Big(\tfrac{1}{2\pi i} \int_l \lambda^{-q/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \dots [\mathcal{D}, a_m] R_s(\lambda) d\lambda\Big) ds.$$

In particular the sum on the right-hand side of (1) *analytically continues to a deleted neighbourhood of $r = (1-q)/2$ with* at worst *a simple pole at $r = (1-q)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=1,\mathrm{odd}}^{2N-1}$ is a (b, B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r = (1-q)/2$.*

(2) *The index is also the residue of a sum of zeta functions:*

$$\frac{1}{\sqrt{2\pi i}}\operatorname{res}_{r=(1-q)/2}\Big(\sum_{m=1,\text{odd}}^{2N-1}\sum_{|k|=0}^{2N-1-m}\sum_{j=0}^{|k|+(m-1)/2}(-1)^{|k|+m}\alpha(k)\Gamma((m+1)/2)\cdot\tilde{\sigma}_{|k|+(m-1)/2,j}(r-(1-q)/2)^j\operatorname{trace}(u^*[\mathcal{D},u]^{(k_1)}[\mathcal{D},u^*]^{(k_2)}\dots\dots[\mathcal{D},u]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2})\Big).$$

In particular the sum of zeta functions on the right-hand side analytically continues to a deleted neighbourhood of $r=(1-q)/2$ and has at worst *a simple pole at $r=(1-q)/2$.*

(3) *If $(\mathcal{A},\mathcal{H},\mathcal{D})$ also has isolated spectral dimension then*

$$\operatorname{index}(PuP)=\frac{1}{\sqrt{2\pi i}}\sum_m \phi_m(\mathrm{Ch}_m(u)),$$

where, for $a_0,\dots,a_m\in\mathcal{A}$,

$$\begin{aligned}\phi_m(a_0,\dots,a_m)&=\operatorname{res}_{r=(1-q)/2}\phi_m^r(a_0,\dots,a_m)\\&=\sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|}\alpha(k)\sum_{j=0}^{|k|+(m-1)/2}\tilde{\sigma}_{(|k|+(m-1)/2),j}\\&\quad\cdot\tau_j(a_0[\mathcal{D},a_1]^{(k_1)}\dots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}),\end{aligned}$$

and $(\phi_m)_{m=1,\text{odd}}^{2N-1}$ is a (b,B)-cocycle for $\mathcal{A}$. When $[q]=2n$ is even, the term with $m=2N-1$ is zero, and for $m=1,3,\dots,2N-3$, all the top terms with $|k|=2N-1-m$ are zero.

Corollary 6.4. *For $1\le p<2$, the statements in* (3) *of Theorem* 6.3 *are true without the assumption of isolated dimension spectrum.*

For even spectral triples we have

Theorem 6.5 (Even local index formula). *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even QC^∞ spectral triple with spectral dimension $q\ge 1$. Let $N=[\frac{q+1}{2}]$, where $[\,\cdot\,]$ denotes the integer part, and let $p\in\mathcal{A}$ be a self-adjoint projection. Then*

$$(1)\qquad \operatorname{index}(p\mathcal{D}^+p)=\operatorname{res}_{r=(1-q)/2}\Big(\sum_{m=0,\text{even}}^{2N}\phi_m^r(\mathrm{Ch}_m(p))\Big),$$

where for $a_0,\dots,a_m\in\mathcal{A}$, $l=\{a+iv: v\in\mathbb{R}\}$, $0<a<1/2$, $R_s(\lambda)=(\lambda-(1+s^2+\mathcal{D}^2))^{-1}$ and $r>1/2$ we define $\phi_m^r(a_0,a_1,\dots,a_m)$ to be

$$\frac{(m/2)!}{m!}\int_0^\infty 2^{m+1}s^m\operatorname{trace}\Big(\gamma\frac{1}{2\pi i}\int_l\lambda^{-q/2-r}a_0R_s(\lambda)[\mathcal{D},a_1]R_s(\lambda)\dots\dots[\mathcal{D},a_m]R_s(\lambda)\,d\lambda\Big)ds.$$

In particular the sum on the right-hand side of (1) *analytically continues to a deleted neighbourhood of* $r = (1-q)/2$ *with* at worst *a simple pole at* $r = (1-q)/2$. *Moreover, the complex function-valued cochain* $(\phi_m^r)_{m=0,\text{even}}^{2N}$ *is a* (b, B)*-cocycle for* $\mathcal{A}$ *modulo functions holomorphic in a half-plane containing* $r = (1-q)/2$.

(2) *The index* $\operatorname{index}(p\mathcal{D}^+p)$ *is also the residue of a sum of zeta functions:*

$$\operatorname{res}_{r=(1-q)/2}\Big(\sum_{m=0,\text{even}}^{2N}\sum_{|k|=0}^{2N-m}\sum_{j=1}^{|k|+m/2}(-1)^{|k|+m/2}\alpha(k)\tfrac{(m/2)!}{2m!}$$
$$\cdot\,\sigma_{|k|+m/2,j}(r-(1-q)/2)^j\operatorname{trace}(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}[\mathcal{D},p]^{(k_2)}\dots$$
$$\dots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2})\Big)$$

(*for* $m = 0$ *we replace* $(2p-1)$ *by* $2p$). *In particular the sum of zeta functions on the right-hand side analytically continues to a deleted neighbourhood of* $r = (1-q)/2$ *and has* at worst *a simple pole at* $r = (1-q)/2$.

(3) *If* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *also has isolated spectral dimension then*

$$\operatorname{index}(p\mathcal{D}^+p) = \sum_{m=0,\text{even}}^{2N}\phi_m(\mathrm{Ch}_m(p))$$

where for $a_0, \dots, a_m \in \mathcal{A}$ *we have* $\phi_0(a_0) = \operatorname{res}_{r=(1-q)/2}\phi_0^r(a_0) = \tau_{-1}(\gamma a_0)$ *and for* $m \geq 2$

$$\phi_m(a_0,\dots,a_m) = \operatorname{res}_{r=(1-q)/2}\phi_m^r(a_0,\dots,a_m)$$
$$= \sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j}$$
$$\cdot\,\tau_{j-1}(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\dots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}),$$

and $(\phi_m)_{m=0,\text{even}}^{2N}$ *is a* (b, B)*-cocycle for* $\mathcal{A}$. *When* $[q] = 2n+1$ *is odd, the term with* $m = 2N$ *is zero, and for* $m = 0, 2, \dots, 2N-2$, *all the top terms with* $|k| = 2N-m$ *are zero.*

Corollary 6.6. *For* $1 \leq q < 2$, *the statements in* (3) *of Theorem* 6.5 *are true without the assumption of isolated dimension spectrum.*

Proposition 6.7 ([24]). *For a* QC^∞ *finitely summable spectral triple with isolated spectral dimension, the collection of functionals* $(\phi_m)_{m=P,P+2,\dots,2N-P}$, $P = 0$ *or* 1, *defined in part* (3) *of Theorems* 6.3 *and* 6.5 *are* (b, B)*-cocycles. They represent the class of the Chern character, and we call this* (b, B)*-cocycle the* ***residue cocycle***.

We remark that the proof of the (b, B)-cocycle property is straightforward once one verifies that the functionals $(\phi_m^r)_{m=P,P+2,\dots,2N-1}$ of part (1) of the Theorems 6.3 and 6.5 are function-valued (b, B)-cocycles (they depend on the complex variable r) modulo functions holomorphic at the critical point $r = (1-q)/2$. This 'almost' cocycle is called the *resolvent cocycle*, and serves as a replacement for the JLO coycle (see below) when we have a finitely summable spectral triple.

Computing the cocycle given by the local index formula is often much easier than computing the Fredholm module version. Understanding *all* the terms and interpreting what they tell us about the 'geometry' of a spectral triple is a major undertaking, involving the construction of many new examples.

6.3 The JLO cocycle. The JLO cocycle is another representative of the Chern character for finitely summable spectral triples [34]. It may be derived in the even case, from the McKean–Singer formula, while in the odd case one uses the θ-summable spectral flow formula of [48], [18] which we now describe. Let $\mathcal{D}_t = \mathcal{D} + tu[\mathcal{D}, u^*]$ for $u \in \mathcal{A}$ unitary, and then the spectral flow along $\{\mathcal{D}_t\}$ is given by

$$\mathrm{index}(PuP) = \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{trace}(u[\mathcal{D}, u^*]e^{-\mathcal{D}_t^2})\, dt.$$

The derivation of the JLO cocycle from this analytic formula in the odd case is in [48], [18].

However, the principal application of the JLO cocycle is in the study of theta summable spectral triples, where the local index formula is not available. These are essential for infinite dimensional situations, for example in supersymmetric quantum field theory. The JLO cocycle is also the starting point for Connes and Moscovici's original proof of the local index formula [40].

Historically θ-summable Fredholm modules and spectral triples were introduced by Connes in association with the study of entire cyclic cohomology, see [34]. The JLO cocycle was discovered later by Jaffe, Lesniewski and Osterwalder [56]. Connes then proved that it is a representative for the Chern character of a theta summable spectral triple. We now describe this representative explicitly. It is given on even spectral triples by an infinite sequence of cochains $(\mathrm{JLO}_{2k})_{k\geq 0}$ defined by

$$\begin{aligned}&\mathrm{JLO}_{2k}(a_0, a_1, \ldots, a_{2k})\\ &= \int_\Delta \mathrm{trace}(\gamma a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D}, a_1]e^{-t_1\mathcal{D}^2} \ldots e^{-t_{2k-1}\mathcal{D}^2}[\mathcal{D}, a_{2k}]e^{-t_{2k}\mathcal{D}^2})\, dt_0 dt_1 \ldots dt_{2k}.\end{aligned}$$

Here $\Delta = \{(t_0, t_1, \ldots, t_{2k}) \in \mathbb{R}^{2k+1} : t_j \geq 0,\ t_0 + t_1 + \cdots + t_{2k} = 1\}$ is the standard simplex.

In the odd case we have $(\mathrm{JLO}_{2k+1})_{k\geq 0}$ defined by

$$\begin{aligned}\mathrm{JLO}_{2k+1}(a_0, a_1, ..., a_{2k+1}) = \sqrt{2\pi i} \int_\Delta \mathrm{trace}(a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D}, a_1]e^{-t_1\mathcal{D}^2} \ldots\\ \ldots e^{-t_{2k}\mathcal{D}^2}[\mathcal{D}, a_{2k}]e^{-t_{2k+1}\mathcal{D}^2})\, dt_0 dt_1 \ldots dt_{2k+1}.\end{aligned}$$

In the context of entire cyclic cohomology, the JLO cocycle represents the Chern character. That is, if $[p] \in K_0(\mathcal{A})$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable, then

$$\langle [p], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = \langle [\mathrm{Ch}(p)], [\mathrm{JLO}(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = \sum_{k=0}^{\infty} \mathrm{JLO}_{2k}(\mathrm{Ch}_{2k}(p)).$$

A similar statement holds in the odd case. The problem with the JLO cocycle is that it is difficult to compute with. For finitely summable spectral triples the local index formula is superior in this regard.

Example 21. Block and Fox showed, [11], starting with the JLO cocycle and using Getzler scaling, that the Chern character for the Dirac operator on a compact spin manifold M can be represented by

$$\mathrm{Ch}_k(f_0, f_1 \dots, f_k) = c_k \int_M \hat{A} f_0\, df_1 \wedge \dots df_k, \quad f_j \in C^\infty(M),$$

where $\hat{A}$ is the A-roof class of the manifold M. This is the Atiyah–Singer index theorem for Dirac operators, from which the general statement can be deduced. Similarly, Ponge used Getzler scaling and Greiner's analysis of heat kernel asymptotics, [52], to obtain the same formula starting from the residue cocycle for $\mathcal{D}$, [75].

Example 22. We explain the local index cocycle for the noncommutative torus. From known computations of the cyclic cohomology of the noncommutative torus, [32], the cocycle arising from the local index formula must be a linear combination of the 0-cocycle τ_0 and the 2-cocycle τ_2 given by

$$\tau_0(a_0) = \tau(a_0), \quad \tau_2(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).$$

Exercise. What is the linear combination? *Hint*: The index pairing with any projection is an integer. Consider $1 \in A_\theta$ and the Powers–Rieffel projector p_θ; see for example [44]. What integers do we expect?

The reason it is worth expressing the index pairing in cyclic theory is highlighted by the last exercise. Cyclic cohomology can often be computed using homological algebra and the relation to Hochschild cohomology. Moreover, there are other operators on these homology theories, and relations between them reminiscent of, for example, Hodge decompositions and the relation between Lie derivatives and the exterior derivative, [34]. All these tools make computations in cyclic theory more practical. However, this is a big topic and beyond the scope of these notes.

7 Semifinite spectral triples and beyond

7.1 Semifinite spectral triples. Applications of noncommutative geometry to number theory and physics require a generalisation of the notion of spectral triple. In this section we discuss one such generalisation. We have already seen that the discussion of the Chern character relies on the theory of trace ideals in $\mathcal{B}(\mathcal{H})$. There are however situations where more general traces and their associated ideals arise [17], [9]. This is the so-called semifinite theory. Thus we begin with some semifinite versions of the standard $\mathcal{B}(\mathcal{H})$ definitions and results.

Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_{\mathcal{N}}$ be the τ-compact operators in $\mathcal{N}$ (that is, the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 7.1. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$,
2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_{\mathcal{N}}$ for all $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$,
3) the triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$, and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

Along with the notion of τ-compact, we naturally get a notion of τ-Fredholm: $T \in \mathcal{N}$ is τ-Fredholm if and only if T is invertible modulo $\mathcal{K}_{\mathcal{N}}$. The index of such operators is in general real-valued, but we can often constrain the possible values. Index pairings with K-theory still make sense, and we are still interested in computing such pairings.

Observe that while we can define a semifinite Fredholm module in a similar way, it is not at all clear at this point what the relation to K-homology is, if any.

Definition 7.2. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.

Remark. The comments made about derivations and domains after Definition 5.1 are just as relevant here. One additional feature is that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1 + \mathcal{D}^2)^{1/2}, T]$. The proofs can be found in [22].

7.1.1 Non-unitality. The examples coming from graph algebras, described soon, are often non-unital. Here is a brief summary of what we require in this case. See [79], [80] and [47] for more information. Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [47], [79], [80], we make two definitions to address these issues.

Definition 7.3. An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$ there exists $\phi \in \mathcal{A}$ such that

$$\phi a_i = a_i \phi = a_i \quad \text{for each } i.$$

Definition 7.4. Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood). If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c \subset \mathcal{A}$ local.

Separable quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n$ for all n, [79]; we call this a local approximate unit. We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra in the following sense.

Definition 7.5. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.

Definition 7.6. A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying

$$\Omega^*_{\mathcal{D}}(\mathcal{A}_c) = \bigcup_n \Omega^*_{\mathcal{D}}(\mathcal{A})_n,$$

where

$$\Omega^*_{\mathcal{D}}(\mathcal{A})_n = \{\omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}) : \phi_n\omega = \omega\phi_n = \omega\}.$$

Remark. A local spectral triple has a local approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n\phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$, see [79], [80]. We require this property to prove the summability results we require.

7.1.2 Semifinite summability. In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [46] that if $S \in \mathcal{N}$, the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\|SE\| : E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)\,dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ need not be complete in this norm but it is complete in the norm $\|\cdot\|_1 + \|\cdot\|_\infty$ (where $\|\cdot\|_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier traces:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \{T \in \mathcal{N} : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \tfrac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds < \infty\}.$$

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from the context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$, [46]. Our notation is however consistent with that of [34] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

Definition 7.7. A semifinite local spectral triple is

- finitely summable if there is some $s_0 \in [0, \infty)$ such that for all $s > s_0$ we have
$$\tau(a(1+\mathcal{D}^2)^{-s/2}) < \infty \quad \text{for all } a \in \mathcal{A}_c;$$
- (p, ∞)-summable if
$$a(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(p,\infty)} \quad \text{for all } a \in \mathcal{A}_c,\ \lambda \in \mathbb{C} \setminus \mathbb{R};$$
- θ-summable if for all $t > 0$ we have
$$\tau(ae^{-t\mathcal{D}^2}) < \infty \quad \text{for all } a \in \mathcal{A}_c.$$

Remark. If $\mathcal{A}$ is unital, and $(1+\mathcal{D}^2)^{-1}$ is τ-compact, ker $\mathcal{D}$ is τ-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the non-unital case.

We need to briefly discuss Dixmier traces in the von Neumann setting, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [25]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds$$

is bounded. For certain elements [25], $\omega \in L^\infty(\mathbb{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is a Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$ and vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $⨍$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbb{R}_*^+)^*$, we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

Example. Let $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ act on $L^2(S^1)$. Then the spectrum of $\mathcal{D}$ consists of eigenvalues $\{n \in \mathbb{Z}\}$, each with multiplicity one. So, using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-1/2}}$ is

$$N \to \frac{1}{\log 2N+1} \sum_{n=-N}^{N} (1+n^2)^{-1/2},$$

which is bounded. So $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and for any Dixmier trace trace_ω we have
$$\mathrm{trace}_\omega((1 + \mathcal{D}^2)^{-1/2}) = ⨍ (1 + \mathcal{D}^2)^{-1/2} = 2.$$

In [79], [80] we proved numerous properties of local algebras. The introduction of quasi-local algebras in [47] led us to review the validity of many of these results for quasi-local algebras. Most of the summability results of [79] are valid in the quasi-local setting. In addition, the summability results of [80] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \geq 1$, [34], [25], and the trace property. We quote the version of the summability results from [80] that we require below, stated just for $p = 1$. This is a non-unital analogue of a result from [25].

Proposition 7.8 ([80]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞, local $(1, \infty)$-summable semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*
$$T(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}.$$
For $\mathrm{Re}(s) > 1$, $T(1 + \mathcal{D}^2)^{-s/2}$ is trace class. If the limit
$$\lim_{s\to 1/2^+} (s - 1/2)\tau(T(1 + \mathcal{D}^2)^{-s})$$
exists, then it is equal to
$$\frac{1}{2} ⨍ T(1 + \mathcal{D}^2)^{-1/2}.$$
In addition, for any Dixmier trace τ_ω, the function
$$a \mapsto \tau_\omega(a(1 + \mathcal{D}^2)^{-1/2})$$
defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.

The various analytic formulae for computing the index, the local index formula, JLO cocycle, Chern characters and the results connecting them all continue to hold in the semifinite case, [17], [18], [21], [22], [23], [24], [25]. Just replace the operator trace in the statement by a general semifinite trace. In addition, the local index formula holds for local spectral triples, semifinite or not, [80].

7.2 Graph algebras. Our aim is to give an example of the application of spectral triples in the noncommutative setting that is quite distinct from any of the classical examples discussed so far. This section also illustrates the fact that the semifinite theory is needed for many cases. Graph algebras and their higher dimensional analogues (k-graphs) are quite a diverse zoo. One can find algebras in these classes (and their relatives such as topological graph algebras, Cuntz–Krieger algebras, etc) with almost any required property. This makes them a great laboratory. They also arise in applications to number theory [15]. The following account of semifinite spectral triples for graph algebras comes from [70].

For a more detailed introduction to graph C*-algebras we refer the reader to [7], [62] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s\colon E^1 \to E^0$ identifying the range and source of each edge. We will always assume that the graph is *row-finite* which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite*, which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2 \ldots \mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \le i < n$. The maps r, s extend to $E^* := \bigcup_{n\ge 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$; we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \ge w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from V). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz–Krieger E-family* in a C*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz–Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \quad \text{and} \quad p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \quad \text{whenever } v \text{ is not a sink.}$$

It is proved in [62], Theorem 1.2, that there is a universal C*-algebra $C^*(E)$ generated by a non-zero Cuntz–Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2 \ldots \mu_n$ is a path in E^n. Since the Cuntz–Krieger relations imply that the projections $S_e S_e^*$ are also mutually orthogonal, we have $S_e^* S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [62], Lemma 1.1). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{7.1}$$

The algebraic relations and the density of $\operatorname{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role. We adopt the conventions that vertices are paths of length 0, that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (7.1) are non-empty; we recover S_μ, for example, by taking $\nu = r(\mu)$, so that $S_\mu S_\nu^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{zS_e, p_v\}$ is another Cuntz–Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z\colon C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\bar z}$ is an inverse for γ_z, so $\gamma_z \in \mathcal{A}ut\, C^*(E)$, and a routine $\varepsilon/3$ argument using (7.1) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action*. Because S^1 is compact, averaging over γ with respect to the normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi} \int_{S^1} \gamma_z(a)\, d\theta \quad \text{for } a \in C^*(E),\ z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From eq. (7.1), it is easy to see that a graph C*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, we always obtain a quasi-local algebra.

Example. For a graph C*-algebra $A = C^*(E)$, eq. (7.1) shows that

$$A_c = \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^*, & v = s(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Similar comments apply to right multiplication by $p_{s(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that A_c has local units.

7.3 Graph C*-algebras with semifinite graph traces. This section considers the existence of (unbounded) traces on graph algebras. We denote by A^+ the positive cone in a C*-algebra A, and we use extended arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. From [73] we take the definition:

Definition 7.9. A trace on a C*-algebra A is a map $\tau\colon A^+ \to [0, \infty]$ satisfying

1) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A^+$,
2) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A^+$ and $\lambda \geq 0$,
3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

We say that τ is faithful if $\tau(a^*a) = 0 \implies a = 0$, that τ is semifinite[3] if $\{a \in A^+ : \tau(a) < \infty\}$ is norm dense in A^+ (or that τ is densely defined), that τ is lower semicontinuous if whenever $a = \lim_{n\to\infty} a_n$ in norm in A^+ we have $\tau(a) \leq \liminf_{n\to\infty} \tau(a_n)$.

[3]The use of semifinite here is different from the von Neumann setting. For a von Neumann algebra, a trace is semifinite if the domain is weakly dense. For a C*-algebra a trace is semifinite if the domain is norm dense.

We may extend a (semifinite) trace τ by linearity to a linear functional on (a dense subspace of) A. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$.

Lemma 7.10. *Let E be a row-finite directed graph and let $\tau\colon C^*(E) \to \mathbb{C}$ be a semifinite trace. Then the dense subalgebra*

$$A_c := \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$$

is contained in the domain I_τ of τ.

It is convenient to denote by $A = C^*(E)$ and $A_c = \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$.

Lemma 7.11. *Let E be a row-finite directed graph.*
(i) *If $C^*(E)$ has a faithful semifinite trace, then no loop can have an exit.*
(ii) *If $C^*(E)$ has a gauge-invariant, semifinite, lower semicontinuous trace τ, then $\tau \circ \Phi = \tau$ and*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu}\tau(p_{r(\mu)}).$$

In particular, τ is supported on $C^*(\{S_\mu S_\mu^* : \mu \in E^*\})$.

Whilst the condition that no loop has an exit is necessary for the existence of a faithful semifinite trace, it is not sufficient.

One of the advantages of graph C*-algebras is the ability to use both graphical and analytical techniques. There is an analogue of the above discussion of traces in terms of the graph.

Definition 7.12 (cf. [89]). If E is a row-finite directed graph, then a graph trace on E is a function $g\colon E^0 \to \mathbb{R}^+$ such that for any $v \in E^0$ we have

$$g(v) = \sum_{s(e)=v} g(r(e)).$$

If $g(v) \neq 0$ for all $v \in E^0$ we say that g is faithful.

Remark. One can show by induction that if g is a graph trace on a directed graph with no sinks, and $n \geq 1$, then

$$g(v) = \sum_{s(\mu)=v,\, |\mu|=n} g(r(\mu)).$$

For graphs with sinks, we must modify this formula to take into account paths of length less than n which end on sinks. To deal with this more general case we write

$$g(v) = \sum_{s(\mu)=v,\, |\mu|\preceq n} g(r(\mu)) \geq \sum_{s(\mu)=v,\, |\mu|=n} g(r(\mu)),$$

where $|\mu| \preceq n$ means that μ is of length n or is of length less than n and terminates on a sink.

As with traces on $C^*(E)$, it is easy to see that a necessary condition for E to have a faithful graph trace is that no loop has an exit.

Proposition 7.13. *Let E be a row-finite directed graph. Then there is a one-to-one correspondence between faithful graph traces on E and faithful, semifinite, lower semicontinuous, gauge-invariant traces on $C^*(E)$.*

There are several steps in the construction of a spectral triple. We begin in Subsection 7.3.1 by constructing a C*-module. We define an unbounded operator $\mathcal{D}$ on this C*-module as the generator of the gauge action of S^1 on the graph algebra. We show in Section 7.3.2 that $\mathcal{D}$ is a regular self-adjoint operator on the C*-module. We use the phase of $\mathcal{D}$ to construct a Kasparov module.

7.3.1 Building a C*-module. Readers unfamiliar with C*-modules should understand that they share some properties of a Hilbert space, except that the inner product takes values in a C*-algebra F, which acts on (the right of) the module. Consequently they are a special class of Banach spaces. This introduces many subtleties into the theory. Fortunately the examples below are straightforward; if necessary more information can be found in [63], [77].

So, in brief, a right C*-module X for the C*-algebra F is a linear space with an action (on the right) of F and an inner product

$$(\cdot\,|\,\cdot)\colon X \times X \to F$$

linear in the second variable, and satisfying $(x|y)^* = (y|x)$ and that $(x|x) \geq 0$ in the sense of positive elements of F, and $(x|x) = 0$ if and only if $x = 0$. We also require that X is complete for the norm $\|x\|^2 = \|(x|x)\|_F$.

The useful things to know concern operators on these modules which commute with the right action of the C*-algebra. One fact is that *not* all F-linear maps $X \to X$ possess adjoints for the inner product. The collection of adjointable endomorphisms (those with an adjoint) is denoted $\mathrm{End}_F(X)$. The adjointable endomorphisms form a C*-algebra with respect to the adjoint operation and operator norm. Amongst these endomorphisms are the rank one endomorphisms $\Theta_{x,y}$, $x, y \in X$, defined on $z \in X$ by

$$\Theta_{x,y} z := x(y|z)_R.$$

Exercise. What is the adjoint of $\Theta_{x,y}$?

Finite sums of rank one endomorphisms are called finite rank. The finite rank endomorphisms generate a closed ideal in $\mathrm{End}_F(X)$. This ideal is called the ideal of compact endomorphisms and is denoted $\mathrm{End}^0_F(X)$.

Two important things should be noted:

(i) We have a notion of compact, so we have a notion of Fredholm (invertible modulo compacts), and so we have a notion of index. In this case the index is a difference of two

F-modules, and this difference defines an element of $K_0(F)$. See [51] for a thorough discussion.

(ii) This notion of compactness need have nothing whatsoever to do with the compactness of operators on Hilbert space or even the notion of compactness in semifinite von Neumann algebras.

The actual C*-modules we will consider in these notes are not complicated, and thus we do not need to delve deeply into the complications of general C*-module theory. Also, the constructions of this subsection work for any locally finite graph.

Let $A = C^*(E)$, where E is any locally finite directed graph. Let $F = C^*(E)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let A_c, F_c be the dense subalgebras of A, F given by the (finite) linear span of the generators. We make A a right inner product F-module. The right action of F on A is by right multiplication. The inner product is defined by

$$(x|y)_R := \Phi(x^*y) \in F.$$

Here Φ is the canonical expectation. It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an F-valued inner product on A. The requirement $(x|x)_R = 0 \implies x = 0$ follows from the faithfulness of Φ.

Definition 7.14. Define X to be the C*-F-module completion of A for the C*-module norm

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

Define X_c to be the pre-C*-F_c-module with linear space A_c and the inner product $(\cdot|\cdot)_R$.

Remark. Typically, the action of F does not map X_c to itself, so we may only consider X_c as an F_c module. This is a reflection of the fact that F_c and A_c are quasilocal, not local.

The inclusion map $\iota\colon A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^*a)\|_F \le \|a^*a\|_A = \|a\|_A^2.$$

We can also define the gauge action γ on $A \subset X$, and as

$$\begin{aligned}\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F\\ &= \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F\\ &= \|\Phi(\gamma_z(a^*a))\|_F = \|\Phi(a^*a)\|_F = \|a\|_X^2,\end{aligned}$$

for each $z \in S^1$, the action of γ_z is isometric on $A \subset X$ and so extends to a unitary U_z on X. This unitary is F linear, adjointable, and we obtain a strongly continuous action of S^1 on X, which we still denote by γ.

For each $k \in \mathbb{Z}$, the projection onto the k-th spectral subspace for the gauge action defines an operator Φ_k on X by

$$\Phi_k(x) = \frac{1}{2\pi} \int_{S^1} z^{-k} \gamma_z(x) d\theta, \quad z = e^{i\theta},\ x \in X.$$

Observe that on generators we have $\Phi_k(S_\alpha S_\beta^*) = S_\alpha S_\beta^*$ when $|\alpha| - |\beta| = k$, and is zero when $|\alpha| - |\beta| \neq k$. The range of Φ_k is

$$\operatorname{range}(\Phi_k) = \{x \in X : \gamma_z(x) = z^k x \text{ for all } z \in S^1\}.$$

These ranges give us a natural $\mathbb{Z}$-grading of X.

Remark. If E is a finite graph with no loops, then for k sufficiently large there are no paths of length k and so $\Phi_k = 0$. This will obviously simplify many of the convergence issues below.

Lemma 7.15. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k \Phi_l = \delta_{k,l} \Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k \in K} \Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k \in \mathbb{Z}} \Phi_k$ converges to the identity operator on X.*

Corollary 7.16. *Let $x \in X$. Then with $x_k = \Phi_k x$ the sum $\sum_{k \in \mathbb{Z}} x_k$ converges in X to x.*

7.3.2 The Kasparov module. In this subsection we assume that E is locally finite and furthermore has no sources. That is, every vertex receives at least one edge.

Since we have the gauge action defined on X, we may use the generator of this action to define an unbounded operator $\mathcal{D}$. We will not define or study $\mathcal{D}$ from the generator point of view, rather taking a more bare-hands approach. It is easy to check that $\mathcal{D}$ as defined below is the generator of the S^1-action.

The theory of unbounded operators on C*-modules that we require is all contained in Lance's book, [63], Chapters 9, 10. We quote the following definitions (adapted to our situation).

Definition 7.17. Let Y be a right C*-B-module. A densely defined unbounded operator $\mathcal{D}\colon \operatorname{dom} \mathcal{D} \subset Y \to Y$ is a B-linear operator defined on a dense B-submodule $\operatorname{dom} \mathcal{D} \subset Y$. The operator $\mathcal{D}$ is closed if the graph

$$G(\mathcal{D}) = \{(x | \mathcal{D}x)_R : x \in \operatorname{dom} \mathcal{D}\}$$

is a closed submodule of $Y \oplus Y$.

If $\mathcal{D}\colon \operatorname{dom} \mathcal{D} \subset Y \to Y$ is densely defined and unbounded, define a submodule

$$\operatorname{dom} \mathcal{D}^* := \{y \in Y : \text{ there exists } z \in Y \text{ such that} \\ \text{for all } x \in \operatorname{dom} \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \operatorname{dom} \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \operatorname{dom} \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^*\colon \operatorname{dom} \mathcal{D}^* \to Y$, $\mathcal{D}^* y = z$, is well defined, and moreover is closed.

Definition 7.18. Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D}\colon \operatorname{dom}\mathcal{D} \subset Y \to Y$ is symmetric if

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R$$

for all $x, y \in \operatorname{dom}\mathcal{D}$. A symmetric operator $\mathcal{D}$ is self-adjoint if $\operatorname{dom}\mathcal{D} = \operatorname{dom}\mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [63], Chapter 9. With these definitions in hand, we return to our C^*-module X.

Proposition 7.19. *Let X be the right C^*-F-module of Definition 7.14. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \left\{x = \textstyle\sum_{k\in\mathbb{Z}} x_k \in X : \|\sum_{k\in\mathbb{Z}} k^2 (x_k|x_k)_R\| < \infty\right\}.$$

For $x = \sum_{k\in\mathbb{Z}} x_k \in X_{\mathcal{D}}$ define

$$\mathcal{D}x = \sum_{k\in\mathbb{Z}} k x_k.$$

Then $\mathcal{D}\colon X_{\mathcal{D}} \to X$ is a self-adjoint regular operator on X.

Remark. Any $S_\alpha S_\beta^* \in A_c$ is in $X_{\mathcal{D}}$ and

$$\mathcal{D} S_\alpha S_\beta^* = (|\alpha| - |\beta|) S_\alpha S_\beta^*.$$

There is a continuous functional calculus for self-adjoint regular operators, [63], Theorem 10.9, and we use this to obtain spectral projections for $\mathcal{D}$ at the C^*-module level. Let $f_k \in C_c(\mathbb{R})$ be 1 in a small neighbourhood of $k \in \mathbb{Z}$ and zero on $(-\infty, k - 1/2] \cup [k + 1/2, \infty)$. Then it is clear that

$$\Phi_k = f_k(\mathcal{D}).$$

That is, the spectral projections of $\mathcal{D}$ are the same as the projections onto the spectral subspaces of the gauge action.

The next Lemma is the first place where we need our graph to be locally finite and have no sources.

Lemma 7.20. *Assume that the directed graph E is locally finite and has no sources. For all $a \in A$ and $k \in \mathbb{Z}$, $a\Phi_k \in \operatorname{End}_F^0(X)$, the compact endomorphisms of the right F-module X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

Remark. The proof actually shows that for $k \geq 0$

$$\Phi_k = \sum_{|\rho|=k} \Theta^R_{S_\rho, S_\rho},$$

where the sum converges in the strict topology. A similar formula holds for $k < 0$.

Lemma 7.21. *Let E be a locally finite directed graph with no sources. For all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. First let $a = p_v$ for $v \in E^0$. Then the sum

$$R_{v,N} := p_v \sum_{k=-N}^{N} \Phi_k (1+k^2)^{-1/2}$$

is finite rank by Lemma 7.20. We will show that the sequence $\{R_{v,N}\}_{N\geq 0}$ is convergent with respect to the operator norm $\|\cdot\|_{\mathrm{End}}$ of endomorphisms of X. Indeed, assuming that $M > N$ we have

$$\begin{aligned}\|R_{v,N} - R_{v,M}\|_{\mathrm{End}} &= \|p_v \sum_{k=-M}^{-N} \Phi_k (1+k^2)^{-1/2} + p_v \sum_{k=N}^{M} \Phi_k (1+k^2)^{-1/2}\|_{\mathrm{End}} \\ &\leq 2(1+N^2)^{-1/2} \to 0\end{aligned}$$

since the ranges of the $p_v\Phi_k$ are orthogonal for different k. Thus, using the argument from Lemma 7.20, $a(1+\mathcal{D}^2)^{-1/2} \in \mathrm{End}_F^0(X)$. Letting $\{a_i\}$ be a Cauchy sequence from A_c, we have

$$\|a_i(1+\mathcal{D}^2)^{-1/2} - a_j(1+\mathcal{D}^2)^{-1/2}\|_{\mathrm{End}} \leq \|a_i - a_j\|_{\mathrm{End}} = \|a_i - a_j\|_A \to 0$$

since $\|(1+\mathcal{D}^2)^{-1/2}\| \leq 1$. Thus the sequence $a_i(1+\mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1+\mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. □

It eventuates that the previous lemmas show that we have a Kasparov module. This is an extension of the notion of Fredholm module, but now, instead of a Hilbert space, we have a C*-module. As for Fredholm modules and spectral triples, they come in two flavours, even and odd.

Definition 7.22. An *odd Kasparov A-B-module* consists of a countably generated ungraded right B-C*-module E, with $\phi\colon A \to \mathrm{End}_B(E)$ a $*$-homomorphism, together with $P \in \mathrm{End}_B(E)$ such that $a(P-P^*)$, $a(P^2-P)$, $[P,a]$ are all compact endomorphisms. Alternatively, for $V = 2P-1$, $a(V-V^*)$, $a(V^2-1)$, $[V,a]$ are all compact endomorphisms for all $a \in A$. One can modify P to $\widetilde{P}$ so that $\widetilde{P}$ is self-adjoint, $\|\widetilde{P}\| \leq 1$, $a(P-\widetilde{P})$ is compact for all $a \in A$ and the other conditions for P hold with $\widetilde{P}$ in place of P without changing the module E. If P has a spectral gap about 0 (as happens in the cases of interest here) then we may and do assume that $\widetilde{P}$ is in fact a projection without changing the module, E.

An *even Kasparov A-B-module* has, in addition to the above data, a grading by a self-adjoint endomorphism Γ with $\Gamma^2 = 1$ and $\phi(a)\Gamma = \Gamma\phi(a)$, $V\Gamma + \Gamma V = 0$.

Just as suitable equivalence relations turned Fredholm modules into a cohomology theory for C*-algebras, so too there are relations which turn Kasparov A-B-modules into a *bivariant* theory, $\mathrm{KK}^*(A,B)$. This works so that

$$\mathrm{KK}^j(A,\mathbb{C}) = K^j(A),\ K\text{-homology}, \quad \mathrm{KK}^j(\mathbb{C},A) = K_j(A),\ K\text{-theory}.$$

By [61], Lemma 2, Section 7, the pair (ϕ, P) determines a $\mathrm{KK}^1(A, B)$ class, and every class has such a representative. As for Fredholm modules, Kasparov modules have an unbounded version as well.

Definition 7.23. An *odd unbounded Kasparov A-B-module* consists of a countably generated ungraded right B-C*-module E, with $\phi\colon A \to \mathrm{End}_B(E)$ a $*$-homomorphism, together with an unbounded self-adjoint regular operator $\mathcal{D}\colon \mathrm{dom}\,\mathcal{D} \subset E \to E$ such that $[\mathcal{D}, a]$ is bounded for all a in a dense $*$-subalgebra of A and $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of E for all $a \in A$. An *even unbounded Kasparov A-B-module* has, in addition to the previous data, a $\mathbb{Z}_2$-grading with A even and $\mathcal{D}$ odd, as in Definition 7.22.

Now we can state a theorem about graph algebras.

Proposition 7.24. *Assume that the directed graph E is locally finite and has no sources. Let $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then (X, V) defines an odd Kasparov module, and so a class in* $\mathrm{KK}^1(A, F)$.

Proof. We will use the approach of [61], Section 4. We need to show that various operators belong to $\mathrm{End}^0_F(X)$. Notice that $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1 - V^2) = a(1+\mathcal{D}^2)^{-1}$ which is compact from Lemma 7.21 and the boundedness of $(1+\mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V, a]$ is compact for all $a \in A$. First we suppose that $a = a_m$ is homogenous for the $\mathbb{T}^1$-action. Then

$$\begin{aligned}[V, a] &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}\\ &= b_1(1+\mathcal{D}^2)^{-1/2} + Vb_2(1+\mathcal{D}^2)^{-1/2},\end{aligned}$$

where $b_1 = [\mathcal{D}, a] = ma$ and $b_2 = [(1+\mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 7.21 will show that $[V, a]$ is compact for all homogenous a. So consider $[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}$ acting on $x = \sum_{k\in\mathbb{Z}} x_k$. We find

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} [(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2} x_k\\ &\quad= \sum_{k\in\mathbb{Z}} ((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2})(1+k^2)^{-1/2} S_\mu S_\nu^* x_k \qquad (7.2)\\ &\quad= \sum_{k\in\mathbb{Z}} f_{\mu,\nu}(k) S_\mu S_\nu^* \Phi_k x.\end{aligned}$$

The function

$$f_{\mu,\nu}(k) = ((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2})(1+k^2)^{-1/2}$$

goes to 0 as $k \to \pm\infty$, and as the $S_\mu S_\nu^* \Phi_k$ are finite rank with orthogonal ranges, the sum in (7.2) converges in the endomorphism norm, and so converges to a compact endomorphism. For $a \in A_c$ we write a as a finite linear combination of generators

$S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism. Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geq 0} \subset A_c$. Then

$$\|[V, a_i - a_j]\|_{\mathrm{End}} \leq 2\|a_i - a_j\|_{\mathrm{End}} \to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. □

7.4 The gauge spectral triple of a graph algebra. In this section we will construct a semifinite spectral triple for those graph C*-algebras which possess a faithful gauge-invariant trace, τ. Recall from Proposition 7.13 that such traces arise from faithful graph traces.

We will begin with the right F_c-module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that E is locally finite and has no sources. This ensures, by Lemma 7.20, that for all $a \in A$ the endomorphisms $a\Phi_k$ of X are compact endomorphisms.

As in the proof of Proposition 7.13, we define a $\mathbb{C}$-valued inner product on X_c:

$$\langle x, y\rangle := \tau((x|y)_R) = \tau(\Phi(x^*y)) = \tau(x^*y).$$

This inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X,\tau)$ to be the completion of X_c for $\langle\cdot,\cdot\rangle$. We need a few lemmas in order to obtain the ingredients of our spectral triple.

Lemma 7.25. *The C^*-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

Lemma 7.26. *Let $\mathcal{H}$, $\mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then, for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well defined on X_c and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leq \big|\,|\alpha| - |\beta|\,\big|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_\infty = \big|\,|\alpha| - |\beta|\,\big|$.

Corollary 7.27. *The algebra A_c is contained in the smooth domain of the derivation δ where $\delta(T) = [|\mathcal{D}|, T]$ for $T \in \mathcal{B}(\mathcal{H})$. That is,*

$$A_c \subseteq \bigcap_{n\geq 0} \operatorname{dom}\delta^n.$$

Definition 7.28. Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By Lemma 5.5, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.

Lemma 7.29. *If $a \in \mathcal{A}$, then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in F \subset \mathcal{A}$ and $a \in \mathcal{A}$ satisfy $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is A. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

7.4.1 Traces and compactness criteria. We still assume that E is a locally finite graph with no sources and that τ is a faithful semifinite lower semicontinuous gauge-invariant trace on $C^*(E)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. The aim of this subsection will then be to prove the following result.

Theorem 7.30. *Let E be a locally finite graph with no sources, and let τ be a faithful, semifinite, gauge-invariant, lower semicontinuous trace on $C^*(E)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞, $(1, \infty)$-summable, odd, local, semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$). For all $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class. If $v \in E^0$ has no sinks downstream then*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$.

We require the definitions of $\mathcal{N}$ and $\tilde{\tau}$, along with some preliminary results.

Definition 7.31. Let $\mathrm{End}_F^{00}(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (\mathrm{End}_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.

Definition 7.32. Let $T \in \mathcal{N}$ and $\mu \in E^*$. Let $|v|_k =$ the number of paths of length k with range v, and define for $|\mu| \neq 0$

$$\omega_\mu(T) = \langle S_\mu, T S_\mu \rangle + \tfrac{1}{|r(\mu)|_{|\mu|}} \langle S_\mu^*, T S_\mu^* \rangle.$$

For $|\mu| = 0$, $S_\mu = p_v$, for some $v \in E^0$, set $\omega_\mu(T) = \langle S_\mu, T S_\mu \rangle$. Define

$$\tilde{\tau} : \mathcal{N}_+ \to [0, \infty] \quad \text{by} \quad \tilde{\tau}(T) = \lim_{L \nearrow} \sum_{\mu \in L \subset E^*} \omega_\mu(T),$$

where L is in the net of finite subsets of E^*.

Remark. For $T, S \in \mathcal{N}_+$ and $\lambda \geq 0$ we have

$$\tilde{\tau}(T + S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda \tilde{\tau}(T) \quad \text{where } 0 \cdot \infty = 0.$$

Proposition 7.33. *The function $\tilde{\tau} : \mathcal{N}_+ \to [0, \infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$\mathrm{End}_F^{00}(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \mathrm{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, *and*

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x\rangle = \tau(y^*x), \quad x, y \in X_c.$$

Notation. If $g\colon E^0 \to \mathbb{R}_+$ is a faithful graph trace, we shall write τ_g for the associated semifinite trace on $C^*(E)$, and $\tilde{\tau}_g$ for the associated faithful, semifinite, normal trace on $\mathcal{N}$ constructed above.

Lemma 7.34. *Let E be a locally finite graph with no sources and a faithful graph trace g. Let $v \in E^0$ and $k \in \mathbb{Z}$. Then*

$$\tilde{\tau}_g(p_v\Phi_k) \le \tau_g(p_v),$$

with equality when $k \le 0$ or when $k > 0$ and there are no sinks within k vertices of v.

Proposition 7.35. *Assume that the directed graph E is locally finite, has no sources and has a faithful graph trace g. For all $a \in A_c$ the operator $a(1+\mathcal{D}^2)^{-1/2}$ is in the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau}_g)$.*

Remark. Using Proposition 7.8, one can check that

$$\operatorname{res}_{s=0} \tilde{\tau}_g(p_v(1+\mathcal{D}^2)^{-1/2-s}) = \tfrac{1}{2}\tilde{\tau}_{g\omega}(p_v(1+\mathcal{D}^2)^{-1/2}). \tag{7.3}$$

We will require this formula when we apply the local index formula.

Corollary 7.36. *Assume E is locally finite, has no sources and has a faithful graph trace g. Then for all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}}$.*

7.5 The index pairing. Having constructed semifinite spectral triples for graph C*-algebras arising from locally finite graphs with no sources and a faithful graph trace, we can apply the semifinite local index formula described in [22]. See also [23], [40], [53].

There is a C*-module index, which takes its values in the K-theory of the core. The numerical index is obtained by applying the trace $\tilde{\tau}$ to the difference of projections representing the K-theory class coming from the C*-module index.

Thus, for any unitary u in a matrix algebra over the graph algebra A,

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle \in \tilde{\tau}_*(K_0(F)).$$

We compute this pairing for unitaries arising from loops (with no exit), which provide a set of generators of $K_1(\mathcal{A})$. To describe the K-theory of the graphs we are considering, we employ the notion of ends.

Definition 7.37. Let E be a row-finite directed graph. An *end* will mean a sink, a loop without exit or an infinite path with no exits.

Remark. We shall identify an end with the vertices which comprise it. Once on an end (of any sort) the graph trace remains constant.

Lemma 7.38. *Let $C^*(E)$ be a graph C^*-algebra such that no loop in the locally finite graph E has an exit. Then*

$$K_0(C^*(E)) = \mathbb{Z}^{\#\text{ends}}, \quad K_1(C^*(E)) = \mathbb{Z}^{\#\text{loops}}.$$

If $A = C^*(E)$ is non-unital, we will denote by A^+ the algebra obtained by adjoining a unit to A; otherwise we let A^+ denote A.

Definition 7.39. Let E be a locally finite graph such that $C^*(E)$ has a faithful graph trace g. Let L be a loop in E, and denote by $p_1, \ldots, p_n$ the projections associated to the vertices of L and by $S_1, \ldots, S_n$ the partial isometries associated to the edges of L, labelled so that $S_n^* S_n = p_1$ and

$$S_i^* S_i = p_{i+1},\ i = 1, \ldots, n-1, \quad S_i S_i^* = p_i,\ i = 1, \ldots, n.$$

Lemma 7.40. *Let $A = C^*(E)$ be a graph C^*-algebra with faithful graph trace g. For each loop L in E we obtain a unitary in A^+,*

$$u = 1 + S_1 + S_2 + \cdots + S_n - (p_1 + p_2 + \cdots + p_n),$$

whose $K_1(A)$ class does not vanish. Moreover, distinct loops give rise to distinct $K_1(A)$ classes, and we obtain a complete set of generators of $K_1(A)$ in this way.

Proposition 7.41. *Let E be a locally finite graph with no sources and a faithful graph trace g and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ is given on the generators of Lemma 7.40 by*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1).$$

Proof. The semifinite local index formula, [22] provides a general formula for the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In our setting it is given by a 1-cochain

$$\phi_1(a_0, a_1) = \text{res}_{s=0} \sqrt{2\pi i}\, \tilde{\tau}_g(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-1/2-s}),$$

and the pairing (with $P = \chi_{[0,\infty)}(\mathcal{D})$) is given by

$$\text{index}(PuP) = \langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \frac{1}{\sqrt{2\pi i}} \phi_1(u, u^*).$$

Now $[\mathcal{D}, u^*] = -\sum S_i^*$ and $u[\mathcal{D}, u^*] = -\sum_{i=1}^n p_i$. Using eq. (7.3) and Proposition 7.35,

$$\begin{aligned} \text{index}(PuP) &= -\text{res}_{s=0}\, \tilde{\tau}_g\left(\textstyle\sum_{i=1}^n p_i (1 + \mathcal{D}^2)^{-1/2-s}\right) \\ &= -\textstyle\sum_{i=1}^n \tau_g(p_i) = -n\tau_g(p_1), \end{aligned}$$

the last equalities following since all the p_i have equal trace and there are no sinks 'downstream' from any p_i since no loop has an exit. □

Remark. The C*-algebra of the graph consisting of a single edge and single vertex is $C(S^1)$ (we choose Lebesgue measure as our trace, normalised so that $\tau(1) = 1$). For this example, the spectral triple we have constructed is the Dirac triple of the circle, $(C^\infty(S^1), L^2(S^1), \frac{1}{i}\frac{d}{d\theta})$ (as can be seen from Corollary 7.43 below.) The index theorem above gives the correct normalisation for the index pairing on the circle. That is, if we denote by z the unitary coming from the construction of Lemma 7.40 applied to this graph, then $\langle[\bar{z}], (\mathcal{A}, \mathcal{H}, \mathcal{D})\rangle = 1$.

Proposition 7.42. *Let E be a locally finite graph with no sources and a faithful graph trace g, and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ can be computed as follows. Let P be the positive spectral projection for $\mathcal{D}$, and perform the C^* index pairing [57]*

$$K_1(A) \times \mathrm{KK}^1(A, F) \to K_0(F), \quad [u] \times [(X, P)] \to [\ker PuP] - [\mathrm{coker}\, PuP].$$

Then we have

$$\mathrm{index}\, PuP = \tilde{\tau}_g(\ker PuP) - \tilde{\tau}_g(\mathrm{coker}\, PuP) = \tilde{\tau}_{g*}([\ker PuP] - [\mathrm{coker}\, PuP]).$$

Proof. It suffices to prove this on the generators of $K_1(A)$ arising from loops L in E. Let $u = 1 + \sum_i S_i - \sum_i p_i$ be the corresponding unitary in A^+ defined in Lemma 7.40. We will show that $\ker PuP = \{0\}$ and that $\mathrm{coker}\, PuP = \sum_{i=1}^n p_i \Phi_1$. For $a \in PX$ write $a = \sum_{m\geq 1} a_m$. For each $m \geq 1$ write $a_m = \sum_{i=1}^n p_i a_m + (1 - \sum_{i=1}^n p_i) a_m$. Then

$$\begin{aligned} PuPa_m &= P(1 - \textstyle\sum_{i=1}^n p_i + \sum_{i=1}^n S_i) a_m \\ &= P(1 - \textstyle\sum^n p_i + \sum^n S_i)(\sum^n p_i a_m) \\ &\quad + P(1 - \textstyle\sum^n p_i + \sum^n S_i)(1 - \sum^n p_i) a_m \\ &= P \textstyle\sum^n S_i a_m + P(1 - \sum^n p_i) a_m \\ &= \textstyle\sum^n S_i a_m + (1 - \sum^n p_i) a_m. \end{aligned}$$

It is clear from this computation that $PuPa_m \neq 0$ for $a_m \neq 0$.

Now suppose $m \geq 2$. If $\sum_{i=1}^n p_i a_m = a_m$ then $a_m = \lim_N \sum_{k=1}^N S_{\mu_k} S_{\nu_k}^*$ with $|\mu_k| - |\nu_k| = m \geq 2$ and $S_{\mu_{k\,1}} = S_i$ for some i. So we can construct b_{m-1} from a_m by removing the initial S_i's. Then $a_m = \sum_{i=1}^n S_i b_{m-1}$, and $\sum_{i=1}^n p_i b_{m-1} = b_{m-1}$. For arbitrary a_m, $m \geq 2$, we can write $a_m = \sum_i p_i a_m + (1 - \sum_i p_i) a_m$, and so

$$\begin{aligned} a_m &= \textstyle\sum^n p_i a_m + (1 - \sum^n p_i) a_m \\ &= \textstyle\sum^n S_i b_{m-1} + (1 - \sum^n p_i) a_m \quad \text{and by adding zero} \\ &= \textstyle\sum^n S_i b_{m-1} + (1 - \sum^n p_i) b_{m-1} + (\sum^n S_i + (1 - \sum^n p_i))(1 - \sum^n p_i) a_m \\ &= u b_{m-1} + u(1 - \textstyle\sum^n p_i) a_m \\ &= PuPb_{m-1} + PuP(1 - \textstyle\sum^n p_i) a_m. \end{aligned}$$

Thus PuP maps onto $\sum_{m\geq 2} \Phi_m X$.

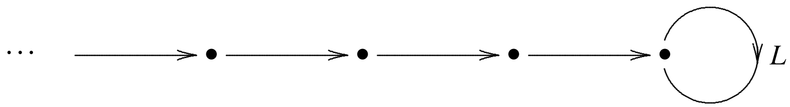
For $m = 1$, if we try to construct b_0 from $\sum_{i=1}^n p_i a_1$ as above, we find $PuPb_0 = 0$ since $Pb_0 = 0$. Thus coker $PuP = \sum^n p_i \Phi_1 X$. By Proposition 7.41, the pairing is then

$$\begin{aligned}\text{index } PuP &= -\sum^n \tau_g(p_i) = -\tilde{\tau}_g\left(\sum^n p_i \Phi_1\right)\\ &= -\tilde{\tau}_{g*}([\text{coker } PuP]) = -\tilde{\tau}_g(\text{coker } PuP).\end{aligned}$$

Thus we can recover the numerical index using $\tilde{\tau}_g$ and the C*-index. □

The following example shows that the semifinite index provides finer invariants of directed graphs than those obtained from the ordinary index. The ordinary index computes the pairing between the K-theory and K-homology of $C^*(E)$, while the semifinite index also depends on the core and the gauge action.

Corollary 7.43 (Example). *Let $C^*(E_n)$ be the algebra determined by the graph*



where the loop L has n edges. Then $C^*(E_n) \cong C(S^1) \otimes \mathcal{K}$ for all n, but n is an invariant of the pair of algebras $(C^*(E_n), F_n)$ where F_n is the core of $C^*(E_n)$.

Proof. Observe that the graph E_n has a one-parameter family of faithful graph traces, specified by $g(v) = r \in \mathbb{R}_+$ for all $v \in E^0$. First consider the case where the graph consists only of the loop L. The C*-algebra A of this graph is isomorphic to $M_n(C(S^1))$, via

$$S_i \to e_{i,i+1},\ i = 1, \ldots, n-1, \quad S_n \to \text{id}_{S^1} e_{n,1},$$

where the $e_{i,j}$ are the standard matrix units for $M_n(\mathbb{C})$. The unitary

$$S_1 S_2 \ldots S_n + S_2 S_3 \ldots S_1 + \cdots + S_n S_1 \ldots S_{n-1}$$

is mapped to the orthogonal sum $\text{id}_{S^1} e_{1,1} \oplus \text{id}_{S^1} e_{2,2} \oplus \cdots \oplus \text{id}_{S^1} e_{n,n}$. The core F of A is $\mathbb{C}^n = \mathbb{C}[p_1, \ldots, p_n]$. Since $\mathrm{KK}^1(A, F)$ is equal to

$$\bigoplus^n \mathrm{KK}^1(A, \mathbb{C}) = \bigoplus^n \mathrm{KK}^1(M_n(C(S^1)), \mathbb{C}) = \bigoplus^n K^1(C(S^1)) = \mathbb{Z}^n,$$

we see that n is the rank of $\mathrm{KK}^1(A, F)$ and so an invariant, but let us link this to the index computed in Propositions 7.41 and 7.42 more explicitly. Let $\phi\colon C(S^1) \to A$ be given by $\phi(\text{id}_{S^1}) = S_1 S_2 \ldots S_n \oplus \sum_{i=2}^n e_{i,i}$. We observe that $\mathcal{D} = \sum_{i=1}^n p_i \mathcal{D}$ because the 'off-diagonal' terms are $p_i \mathcal{D} p_j = \mathcal{D} p_i p_j = 0$. Since $S_1 S_1^* = S_n^* S_n = p_1$, we find (with P the positive spectral projection of $\mathcal{D}$)

$$\phi^*(X, P) = (p_1 X, p_1 P p_1) \oplus \text{degenerate module} \in \mathrm{KK}^1(C(S^1), F).$$

Now let $\psi: F \to \mathbb{C}^n$ be given by $\psi(\sum_j z_j p_j) = (z_1, z_2, \ldots, z_n)$. Then

$$\psi_*\phi^*(X, P) = \bigoplus_{j=1}^{n} (p_1 X p_j, p_1 P p_1) \in \bigoplus_{j=1}^{n} K^1(C(S^1)).$$

Now $X \cong M_n(C(S^1))$, so $p_1 X p_j \cong C(S^1)$ for each $j = 1, \ldots, n$. It is easy to check that $p_1 \mathcal{D} p_1$ acts by $\frac{1}{i}\frac{d}{d\theta}$ on each $p_1 X p_j$, and so our Kasparov module maps to

$$\psi_*\phi^*(X, P) = \bigoplus_{j=1}^{n} (C(S^1), P_{\frac{1}{i}\frac{d}{d\theta}}) \in \oplus^n K^1(C(S^1)),$$

where $P_{\frac{1}{i}\frac{d}{d\theta}}$ is the positive spectral projection of $\frac{1}{i}\frac{d}{d\theta}$. The pairing with id_{S^1} is nontrivial on each summand, since $\phi(\mathrm{id}_{S^1}) = S_1 \ldots S_n \oplus \sum_{i=2}^{n} e_{i,i}$ is a unitary mapping $p_1 X p_j$ to itself for each j. So we have, [54],

$$\begin{aligned} \mathrm{id}_{S^1} \times \psi_*\phi^*(X, P) &= \sum_{j=1}^{n} \mathrm{index}(P \mathrm{id}_{S^1} P : p_1 P X p_j \to p_1 P X p_j) \\ &= -\sum_{j=1}^{n} [p_j] \in K_0(\mathbb{C}^n). \end{aligned}$$

By Proposition 7.42, applying the trace to this index gives $-n\tau_g(p_1)$. Of course in Proposition 7.42 we used the unitary $S_1 + S_2 + \cdots + S_n$; however in $K_1(A)$

$$[S_1 S_2 \ldots S_n] = [S_1 + S_2 + \cdots + S_n] = [\mathrm{id}_{S^1}].$$

To see this, observe that

$$(S_1 + \cdots + S_n)^n = S_1 S_2 \ldots S_n + S_2 S_3 \ldots S_1 + \cdots + S_n S_1 \ldots S_{n-1}.$$

This is the orthogonal sum of n copies of id_{S^1}, which is equivalent in K_1 to $n[\mathrm{id}_{S^1}]$. Finally, $[S_1 + \cdots + S_n] = [\mathrm{id}_{S^1}]$ and so

$$[(S_1 + \cdots + S_n)^n] = n[S_1 + \cdots + S_n] = n[\mathrm{id}_{S^1}].$$

Since we have cancellation in K_1, this implies that the class of $S_1 + \cdots + S_n$ coincides with the class of $S_1 S_2 \ldots S_n$.

Having seen what is involved, we now add the infinite path on the left. The core becomes $\mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ (n copies). Since $A = C(S^1) \otimes \mathcal{K} = M_n(C(S^1)) \otimes \mathcal{K}$, the intrepid reader can go through the details of an argument similar to the one above, with entirely analogous results. □

The invariants obtained from the semifinite index are finer than the isomorphism class of $C^*(E)$, depending as they do on $C^*(E)$ and the gauge action, or equivalently $C^*(E)$ and F.

7.6 The relationship between semifinite triples and KK-theory. In order to construct a semifinite spectral triple for a graph algebra with gauge-invariant trace, we first constructed a Kasparov module. The numerical index we computed was then compatible with the Kasparov product (K-theory-valued index). The question is whether this is always the case. The following proposition from [57] gives an affirmative answer. While stated for unital algebras, it can be generalised to the non-unital setting.

Proposition 7.44. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a unital semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Suppose that the norm closure $A = \bar{\mathcal{A}}$ of $\mathcal{A}$ is a separable C^*-algebra. Let $P = \chi_{[0,\infty)}(\mathcal{D})$ and let $u \in \mathcal{A}$ be unitary. For V a closed subspace of $\mathcal{H}$ we write Q_V for the orthogonal projection onto V. Set $\mathcal{D}_t = (1-t)\mathcal{D} + tu^*\mathcal{D}u = \mathcal{D} + tu^*[\mathcal{D}, u]$; then the unbounded semifinite spectral flow of the path $t \mapsto \mathcal{D}_t$ is given by*

$$\mathrm{sf}\{\mathcal{D}_t\} = \mathrm{index}(PuP) = \tau_*\big(Q_{\ker(pup+1-p)}\big) - \tau_*(Q_{\ker(pu^*p+1-p)}),$$

where $\tau_* \colon K_0(\mathcal{K}_{\mathcal{N}}) \to \mathbb{R}$ is the homomorphism induced by the trace τ and $p = \chi_{[0,\infty)}(F_{\mathcal{D}})$. In addition there exists a separable C^*-algebra $B \subseteq \mathcal{K}_{\mathcal{N}}$ and a class $[\mathcal{D}_B] \in \mathrm{KK}^1(A, B)$ such that

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau(i_*([u] \otimes_A [\mathcal{D}_B])),$$

where $i \colon B \to \mathcal{K}_N$ is the inclusion and $[u] \in K_1(A)$ is the class of the unitary.

Thus semifinite index theory is a special, computable, case of Kasparov theory. The greater the constraint we can place on the 'right-hand' algebra B, the more constraint we place on the possible values of the index. Since the index is a priori any real number, this can be very important.

For the graph of the previous section, the index actually tells us the value of the graph trace on a projection (analytic input), and the number of vertices on the loop (topological data).

7.7 Modular spectral triples, type III von Neumann algebras and KMS states. The examples of spectral triples associated to graph algebras that we have discussed to this point are not sufficient for all applications. To illustrate this point: the Cuntz algebras O_n, $n = 2, 3, \dots,$ are graph algebras. Following the methods described previously enables us to construct a Kasparov module. However, O_n has no nontrivial traces, so we cannot construct a semifinite spectral triple using the methods we employed previously. The Cuntz algebras do however admit twisted traces. By this we mean there are densely defined non-$*$-automorphisms σ and states ϕ on O_n such that $\phi(ab) = \phi(\sigma(b)a)$ for all a, b in a dense subalgebra. These twisted traces are in fact KMS states for certain actions of the reals (the theory of which is briefly explained on p. 57 in this volume). It is a subject of ongoing research at the moment to understand how to construct an index theory for such KMS states. The idea which has been expounded in [20] is to modify the definition of spectral triple so as to accommodate the twist. Because the von Neumann algebra generated by the algebra in the GNS representation associated to a twisted trace may be type III there is no elementary way to do this. However, there is a proposal that gives interesting results for a variety of examples, including the Cuntz algebras, [20], the algebra $SU_q(2)$, [27] and the graph algebras that arise in number theory [42]. Also, a class of examples extending the Cuntz algebra examples, constructed using a topological version of the group measure space construction, is contained in [19].

Recently there has been substantial progress for the general situation where one studies KMS states for periodic actions of the reals on general C^*-algebras, [16]. In this framework a general index theorem has been proved. There is an application of this KMS point of view to number theory in the form of constructing invariants of Mumford curves [15]. A discussion would however take us too far afield. Nevertheless [16], [15] indicate that there are connections between ideas from quantum statistical mechanics in the form of KMS states, equivariant KK-theory and the geometry of singular spaces.

Appendix: Unbounded operators on Hilbert space

This appendix is based primarily on [54], but also see [78].

Definition A.1. An unbounded operator D on a Hilbert space $\mathcal{H}$ is a linear map from a subspace $\operatorname{dom} D \subset \mathcal{H}$ (called the domain of D) to $\mathcal{H}$. The unbounded operator D is said to be densely defined if $\operatorname{dom} D$ is dense in $\mathcal{H}$.

We are really only interested in densely defined operators.

Definition A.2. If D, D' are unbounded operators on $\mathcal{H}$ and $\operatorname{dom} D \subset \operatorname{dom} D'$ and $D\xi = D'\xi$ for all $\xi \in \operatorname{dom} D$, then we write $D \subseteq D'$ and say that D' is an extension of D.

Definition A.3. If D is an (unbounded) operator on $\mathcal{H}$, the graph of D is the subspace $\{(\xi, D\xi) : \xi \in \operatorname{dom} D\} \subset \mathcal{H} \times \mathcal{H}$. The operator D is said to be closed if the graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. The operator D is said to be closable if D has a closed extension D'.

If $\operatorname{dom} D$ is all of $\mathcal{H}$ and D is closed, then the closed graph theorem shows that D is bounded. For an unbounded operator D to be closed we must have: whenever $\{\xi_k\}_{k\geq 1} \subset \operatorname{dom} D$ is a convergent sequence such that $\{D\xi_k\}_{k\geq 1}$ is also a convergent sequence we have $\lim_{k\to\infty} D\xi_k = D \lim_{k\to\infty} \xi_k$.

Any closable operator has a closure $\bar{D} \supseteq D$ which is the operator whose graph is the closure of the graph of D.

Definition A.4. Let D be an unbounded densely defined operator on $\mathcal{H}$. Define

$$\operatorname{dom} D^* = \{\eta \in \mathcal{H} : \text{for all } \xi \in \operatorname{dom} D \text{ there exists } \rho \in \mathcal{H} \text{ such that } \langle D\xi, \eta\rangle = \langle \xi, \rho\rangle\}.$$

Then we define $D^*\colon \operatorname{dom} D^* \to \mathcal{H}$ by $D^*\eta = \rho$. This is well defined, and the operator D^* is closed.

Exercise. Prove the two assertions of the definition.

Definition A.5. An operator D is symmetric if $D \subseteq D^*$, so

$$\langle D\xi, \eta\rangle = \langle \xi, D\eta\rangle \quad \text{for all } \xi, \eta \in \operatorname{dom} D.$$

The operator D is self-adjoint if $D = D^*$, so D is symmetric and $\operatorname{dom} D = \operatorname{dom} D^*$.

Despite appearances, there is a world of difference between symmetric and self-adjoint operators. If D is symmetric then it is closable and $D \subseteq \bar{D} \subseteq D^*$. If $\operatorname{dom} \bar{D} = \operatorname{dom} D^*$ then we say that D is essentially self-adjoint, meaning it has a unique self-adjoint extension.

Let D be a closed operator, and give $\operatorname{dom} D$ the graph norm

$$\|\xi\|_D^2 = \|\xi\|^2 + \|D\xi\|^2.$$

Then $\operatorname{dom} D$ is closed in the topology coming from the graph norm. The resolvent set of D is the set of all $\lambda \in \mathbb{C}$ such that the operator

$$(D - \lambda \mathrm{id}_{\mathcal{H}})\colon \operatorname{dom} D \to \mathcal{H}$$

has a two-sided inverse. Any such inverse is a bounded operator from $\mathcal{H}$ to $\operatorname{dom} D$ and so is a bounded operator.

The spectrum of D is the complement of the resolvent set, i.e., those $\lambda \in \mathbb{C}$ such that $(D - \lambda \mathrm{id}_{\mathcal{H}})$ is not invertible.

Lemma A.6. *The spectrum of a self-adjoint operator is real.*

This allows us, after some effort, to come up with a functional calculus for self-adjoint operators. This functional calculus allows us to define $f(D)$ for any bounded Borel function on the spectrum of D. If $f_n \to f$ pointwise, then $f_n(D) \to f(D)$ in the strong operator topology. With suitable care with domains, it is also possible to define unbounded Borel functions of D. For a thorough discussion of this, see [78].

Two important results that we exploit in the text are:

(i) Any differential operator on a manifold-without-boundary is closable.

(ii) Every symmetric differential operator on a compact manifold-without-boundary is essentially self-adjoint.

Proofs of these two results can be found in [54].

Finally, an unbounded operator D on a Hilbert space $\mathcal{H}$ is said to be affiliated to a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ if for all projections p in the commutant of $\mathcal{N}$ we have $p\colon \operatorname{dom} D \to \operatorname{dom} D$ and $Dp = pD$.

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